

THEORY AND ALGORITHMS FOR STRUCTURED NONSMOOTH OPTIMIZATION

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Shanshan Zhang

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Shanshan Zhang, Ph.D.

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Nonsmoothness in optimization is typically highly structured, and this structure is fundamental to any understanding of many concrete settings. This dissertation, consisting of two core chapters (followed by some further developments), studies structured nonsmooth optimization from both theoretical and algorithmic perspectives.

In the first central chapter, we consider a broad class of nonsmooth functions called “partly smooth functions”, which are well-behaved relative to a certain smooth manifold and moreover enjoy a powerful calculus and sensitivity theory. We use the generalized Hessian of Mordukhovich to study the second-order conditions and sensitivity theory for this class of optimization problems. In this setting, the generalized Hessian is easy to compute. Using this computation we derive various illuminating characterizations of tilt stability, an influential sensitivity concept for the subdifferential introduced by Poliquin and Rockafellar [38]. We moreover relate this notion to the idea of “strong metric regularity”.

In the second core chapter, we investigate the potential behavior, both good and bad, of the well-known BFGS algorithm for smooth minimization, when applied to nonsmooth functions. We consider three very particular examples. We first present a simple nonsmooth example, illustrating how BFGS (in this case with an exact line search) typically succeeds despite nonsmoothness. We

then explore, computationally, the behavior of the BFGS method with an inexact line search on the same example, and discuss the results. In further support of the heuristic effectiveness of BFGS despite nonsmoothness, we prove that, for the very simplest example of a nonsmooth function (a maximum of two affine functions), the method cannot stall at a nonstationary point. On the other hand, we present a nonsmooth example where the inexact-line-search BFGS method converges to a point despite the presence of directions of linear descent. Finally, we briefly compare line-search and trust-region strategies for BFGS in the nonsmooth case.

A subsequent chapter explores some preliminary related development, and considers open questions.

BIOGRAPHICAL SKETCH

Shanshan Zhang was born in Hekou, a small and beautiful town located in the northeast of Jiangxi, China. After spending the first seventeen years in her hometown, she moved to Beijing to attend Peking University in 2003 and obtained her B.S. in Mathematics in 2008. From there, she came to United States to pursue a doctoral degree in the School of Operations Research and Information Engineering.

Upon graduating from Cornell, Shanshan will join Oracle Corporation in Redwood City, CA, where she will embark on a new journey of her life.

To my parents:
Yibing Zhang and Xiane Zhu

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CHAPTER 1

INTRODUCTION

1.1 Nonsmoothness in classical optimization

Nonsmoothness is prevalent in optimization problems. Contrary to our intuition that nonsmoothness is somehow pathological, it typically arises in a highly structured manner. To be more specific, nonsmoothness in an optimization problem is usually associated with an “active” set locally around an optimal solution. Knowledge of this active set is crucial for deeper understanding of the problem with respect to sensitivity analysis and optimality conditions. This phenomenon of nonsmoothness related to an active set is a recurring topic in optimization.

In classical nonlinear constrained optimization, even though the objective and constraint functions are smooth, nonsmoothness exists in the geometry of the feasible region. Let's consider the following standard nonlinear programming problem.

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & f(x) \\ \text{subject to} \quad & c_i(x) = 0, i \in \mathcal{E}, \\ & c_j(x) \geq 0, j \in \mathcal{I}, \end{aligned} \tag{1.1}$$

where the objective function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ and the constraint functions $c_i : \mathbf{R}^n \rightarrow \mathbf{R}$ are C^2 smooth. The corresponding *Lagrangian function* is defined by

$$L(x, \lambda) = f(x) - \sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i c_i(x).$$

The *active set* for x is defined by

$$\mathcal{A}(x) := \{ i \in \mathcal{I} \cup \mathcal{E} \mid c_i(x) = 0 \}.$$

This active set “captures” the nonsmoothness of the feasible region locally around x , as we will see shortly. The following are the standard first-order necessary conditions for optimality.

Theorem 1.1. [31, Thm 12.1] Suppose x^* is a local solution of the problem (1.1). Assume the linear independence constraint qualification (LICQ) holds at x^* , that is, the set of active constraint gradients $\{ \nabla c_i(x^*), i \in \mathcal{A}(x^*) \}$ is linearly independent. Then there is a Lagrange multiplier vector λ^* , with components λ_i^* , $i \in \mathcal{I} \cup \mathcal{E}$, such that the following conditions are satisfied:

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) &= 0, \\ c_i(x^*) &= 0, & \text{for all } i \in \mathcal{E}, \\ c_i(x^*) &\geq 0, & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, & \text{for all } i \in \mathcal{I}, \\ \lambda_i^* c_i(x^*) &= 0, & \text{for all } i \in \mathcal{I} \cup \mathcal{E}. \end{aligned} \tag{1.2}$$

The conditions (1.2) are known as Karush-Kuhn-Tucker conditions (KKT conditions). We will have occasion to use a constraint qualification weaker than LICQ namely the Mangasarian-Fromovitz constraint qualification (MFCQ): there is a vector $v \in \mathbf{R}^n$ such that

$$\begin{aligned} \nabla c_i(x^*)^T v &> 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I}, \\ \nabla c(x^*)^T v &= 0 & \text{for all } i \in \mathcal{E}, \\ \{ \nabla c_i(x^*), i \in \mathcal{E} \} & \text{ is linearly independent.} \end{aligned} \tag{1.3}$$

The first-order KKT conditions tell us how the first derivatives of the objective and active constraints are interrelated at a locally optimal solution. What can second-order information tell us about the optimization problem? The second-order conditions deal with the curvature of the Lagrangian function along “critical” directions. Given (x^*, λ^*) satisfying the KKT conditions, the *critical cone*

$C(x^*, \lambda^*)$ is defined by

$$w \in C(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^T w = 0 & \text{for all } i \in \mathcal{E}, \\ \nabla c_i(x^*)^T w = 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0, \\ \nabla c_i(x^*)^T w \geq 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0. \end{cases}$$

Intuitively speaking, the critical cone contains directions w along which we could make a small change but still maintain those active constraints corresponding to strictly positive multipliers. The second-order necessary conditions require the Hessian of the Lagrangian to have nonnegative curvature along directions in $C(x^*, \lambda^*)$.

Theorem 1.2. [31, Thm 12.5] *Suppose that x^* is a local solution of the problem (1.1) and that the LICQ condition is satisfied. Let λ^* be the Lagrange multiplier vector for which the KKT conditions (1.2) are satisfied. Then*

$$w^T \nabla_{xx}^2 L(x^*, \lambda^*) w \geq 0, \text{ for all } w \in C(x^*, \lambda^*). \quad (1.4)$$

In contrast to first-order information alone, second-order information can yield sufficient conditions for optimality.

Theorem 1.3. [31, Thm 12.6] *Suppose for some feasible solution x^* for the problem (1.1) there is a Lagrange multiplier vector λ^* such that the KKT conditions (1.2) are satisfied. Suppose also that*

$$w^T \nabla_{xx} L(x^*, \lambda^*) w > 0, \text{ for all } w \in C(x^*, \lambda^*), w \neq 0. \quad (1.5)$$

Then x^ is a strict local solution for (1.1).*

A fundamental question in optimization concerns the behavior of a locally optimal solution relative to small perturbations made in the objective and constraints. A general parametric nonlinear programming problem can be stated

as follows.

$$\mathcal{P}(u) \quad \begin{cases} \min_{x \in \mathbf{R}^n} & f(x, u), \\ \text{subject to} & c_i(x, u) = 0, \quad i \in \mathcal{E}, \\ & c_j(x, u) \geq 0, \quad j \in \mathcal{I}, \end{cases} \quad (1.6)$$

where the objective function $f(x, u) : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$ and constraint functions $c_i(x, u) : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$ are C^2 smooth. Under reasonable assumptions, there exists a locally optimal solution $x(u)$ for the problem $\mathcal{P}(u)$ depending Lipschitz continuously on u and such that the active set for $x(u)$ is stable.

Theorem 1.4. [11, Thm 3.1] *Suppose that the second-order sufficient condition for a local minimum for the problem $\mathcal{P}(u^*)$ hold at x^* with associated Lagrange vector λ^* , that the LICQ condition holds at x^* for $\mathcal{P}(u^*)$, and that the strict complementary slackness condition (SCS) holds at x^* with respect to λ^* for $\mathcal{P}(u^*)$, that is*

$$\lambda_i^* > 0 \text{ whenever } c_i(x^*, u^*) = 0, i \in \mathcal{I}.$$

Then the following conditions hold.

1. x^* is a strict local minimum for $\mathcal{P}(u^*)$ (and in fact the objective function grows quadratically around x^* on the feasible region), and the associated Lagrange vector λ^* is unique;
2. on a neighborhood of u^* , there exists a unique C^1 smooth vector function $y(u) = (x(u), \lambda(u))$ satisfying the second-order sufficient optimality conditions for the problem $\mathcal{P}(u)$ such that $y(u^*) = (x(u^*), \lambda(u^*))$ and, hence $x(u)$ is a locally unique optimal solution of $\mathcal{P}(u)$ with associated unique Lagrange vector $\lambda(u)$;
3. the LICQ and SCS conditions hold at $x(u)$ for u near u^* .

So far, we have described how nonsmoothness induces an active set central to optimality conditions and sensitivity analysis in classical nonlinear programming. Theory about active sets is also fundamental for active-set methods. In

general, active-set methods in classical constrained optimization estimate the active set for the problem, and then solve the KKT conditions approximately, and repeat the process by updating the active set accordingly until optimality conditions are nearly satisfied (see [31]). The following active-set identification theorem guarantees that any point with the KKT conditions nearly satisfied actually lies in the active set, under reasonable assumptions. In what follows the *distance function* between two sets in \mathbf{R}^n , defined by

$$\text{dist}(C_1, C_2) = \inf_{x \in C_1, y \in C_2} \|x - y\|_2.$$

Also, we will describe a threshold test to measure how far the estimate x is away from the solution x^* to (1.1) by using the following function $\psi(x, \lambda)$:

$$\psi(x, \lambda) = \left\| \begin{bmatrix} \nabla_x L(x, \lambda) \\ c_i(x) \\ \min(c_j(x), \lambda_j) \end{bmatrix} \right\|_1,$$

where $i \in \mathcal{E}$ and $j \in \mathcal{I}$. The threshold test for the active set estimate is given as follows for a given parameter $\sigma \in (0, 1)$:

$$\mathcal{A}(x, \lambda) = \{i | c_i(x) \leq \psi(x, \lambda)^\sigma, i \in \mathcal{E} \cup \mathcal{I}\}.$$

Theorem 1.5. [33, Thm 2.2] Suppose that the KKT conditions (1.2) and the MFCQ conditions (1.3) are satisfied at the point x^* . Let \mathcal{S}_D denote the set of possible multipliers:

$$\mathcal{S}_D = \{\lambda^* : (x^*, \lambda^*) \text{ satisfies (1.2)}\}.$$

Moreover, suppose for all $\lambda^* \in \mathcal{S}_D$, the second-order sufficient conditions (1.5) are satisfied. Then there exists an $\epsilon > 0$ such that for all (x, λ) satisfying $\|x - x^*\| \leq \epsilon$, $\lambda_i \geq 0$, ($i \in \mathcal{I}$) and $\text{dist}(\lambda, \mathcal{S}_D) \leq \epsilon$, we have $\mathcal{A}(x, \lambda) = \mathcal{A}(x^*)$.

The idea of active-set methods is prevalent more generally in nonsmooth optimization. As one simple example, for the problem of minimizing the sum

of Euclidean norms considered in [36], nonsmoothness of the objective function is associated with an “active” set that is highly structured. Active-set algorithms for this problem can be found in [36].

To summarize the discussion of nonlinear programming, the constraints of a classical problem define a nonsmooth feasible region that typically has nice structure, associated with an active set around a locally optimal solution; the active set is stable under small perturbation to the problem, under reasonable conditions; furthermore, it is also central in algorithm design.

1.2 A nonsmooth example

We will use a simple nonsmooth example to illustrate how to abstract the geometry of active sets so as to extend classical nonlinear programming theory to more general problems of nonsmooth optimization. Suppose that $\bar{x} \in \mathbf{R}^n$ locally minimizes a pointwise maximum of C^2 smooth functions

$$f(x) = \max_{i \in I} f_i(x), \quad (1.7)$$

with *affinely independent* $\nabla f_i(\bar{x})$ for i in the active set

$$\bar{I} := \{ i : f_i(\bar{x}) = f(\bar{x}) \} = I(\bar{x}).$$

By affine independence we mean that the vectors $\{(\nabla f_i(\bar{x}), -1), i \in \bar{I}\}$ are linearly independent.

First note that this nonsmooth problem is equivalent to the following classical nonlinear programming problem.

$$\begin{aligned} \min_{(x,t) \in \mathbf{R}^n \times \mathbf{R}} \quad & t \\ \text{s.t.} \quad & f_i(x) \leq t, i \in I. \end{aligned} \quad (1.8)$$

Indeed, let $\bar{t} = \max_{i \in \mathcal{I}} f_i(\bar{x})$ and observe that (\bar{x}, \bar{t}) is a local minimizer of problem (1.8). The active set $\mathcal{A}(\bar{x}, \bar{t})$ is the same as $\bar{\mathcal{I}}$ and the LICQ condition holds.

We will use Theorems 1.1 and 1.2 to derive the optimization conditions for (1.8). Since (\bar{x}, \bar{t}) is a local minimizer, there exists unique $\bar{\lambda}$ such that the KKT conditions are satisfied, that is

$$\sum_{i \in \bar{\mathcal{I}}} \bar{\lambda}_i \nabla f_i(\bar{x}) = 0, \quad \sum_{i \in \bar{\mathcal{I}}} \bar{\lambda}_i = 1, \quad \text{and } \bar{\lambda} \geq 0. \quad (1.9)$$

Let's consider the second-order optimality conditions. Given that $(\bar{x}, \bar{t}, \bar{\lambda})$ satisfies the KKT conditions, the critical cone at $(\bar{x}, \bar{\lambda})$ is

$$C(\bar{x}, \bar{t}, \bar{\lambda}) = \left\{ (w, 0) \in \mathbf{R}^n \times \mathbf{R} \left| \begin{array}{l} \nabla f_i(\bar{x})^T w = 0, \quad \bar{\lambda}_i > 0 \text{ and } i \in \bar{\mathcal{I}}, \\ \nabla f_i(\bar{x})^T w \leq 0, \quad \bar{\lambda}_i = 0 \text{ and } i \in \bar{\mathcal{I}}. \end{array} \right. \right\}.$$

Then the necessary second-order optimality conditions amount to

$$w^T \nabla_{xx}^2 L(\bar{x}, \bar{t}, \bar{\lambda}) w = w^T \sum_{i \in \bar{\mathcal{I}}} \bar{\lambda}_i \nabla^2 f_i(\bar{x}) w \geq 0, \text{ for all } w \in C(\bar{x}, \bar{t}, \bar{\lambda}).$$

For the parametric situation, again for illustration, we consider a simple linear perturbation:

$$f(x, u) = \max_{i \in \mathcal{I}} f_i(x) + \langle u, x \rangle.$$

This perturbed problem is equivalent to

$$\mathcal{P}(u) \quad \left\{ \begin{array}{ll} \min_{(x, t) \in \mathbf{R}^n \times \mathbf{R}} & t + \langle u, x \rangle \\ \text{s.t.} & f_i(x) \leq t, i \in \mathcal{I}. \end{array} \right. \quad (1.10)$$

In order to apply Theorem 1.4, we now require that the SCS condition holds at $x(0) := \bar{x}$ with respect to $\lambda(0) := \bar{\lambda}$ for $\mathcal{P}(0)$. Under this extra condition, we can derive that the Lagrange multiplier vector $\bar{\lambda}$ satisfies $\bar{\lambda}_i > 0$ for $i \in \bar{\mathcal{I}}$. Then, the critical cone is reduced to

$$C(\bar{x}, \bar{t}, \bar{\lambda}) = \{ (w, 0) \in \mathbf{R}^n \times \mathbf{R} \mid \nabla f_i(\bar{x})^T w = 0, i \in \bar{\mathcal{I}} \}.$$

If the second-order sufficient conditions hold, i.e.

$$w^T \sum_{i \in \bar{I}} \bar{\lambda}_i \nabla^2 f_i(\bar{x}) w > 0 \text{ for any } 0 \neq w \in C(\bar{x}, \bar{t}, \bar{\lambda}),$$

then for all small u the minimizer $x(u)$ of $\mathcal{P}(u)$ depends Lipschitz continuously on u , the associated Lagrange vector $\lambda(u)$ is unique, and the LICQ and SCS conditions hold at $x(u)$.

To summarize, we started with a nonsmooth example and used the equivalent nonlinear programming problem to derive optimality conditions and sensitivity information. A more natural idea is to use the “essential geometry” of the problem to study these objects. We define the *active manifold*

$$\mathcal{M} = \{x : \mathcal{I}(x) = \bar{I}\}.$$

In this example, the *normal space* $N_{\mathcal{M}}(\bar{x})$ is defined by

$$\text{span} \{\nabla f_i(\bar{x}) \mid i \in \bar{I}\},$$

and the *tangent space* $T_{\mathcal{M}}(\bar{x})$ is defined by

$$\{w \mid \nabla f_i(\bar{x})^T w = 0, i \in \bar{I}\}.$$

The subdifferential of a general nonsmooth function is analogous to the gradient of a smooth function. We will give a precise definition later. In this particular example, the subdifferential is the convex hull of limits of gradients at nearby points, or more precisely

$$\partial f(\bar{x}) = \text{conv} \{ \lim \nabla f(x_n) : x_n \rightarrow \bar{x} \}.$$

Clearly this is equivalent to

$$\partial f(\bar{x}) = \left\{ \sum_{i \in \bar{I}} \lambda_i \nabla f_i(\bar{x}) : \lambda_i \geq 0, \sum_{i \in \bar{I}} \lambda_i = 1 \right\}.$$

The function f is “partly smooth” relative to the manifold \mathcal{M} in the following sense.

1. The function f is smooth relative to the active manifold \mathcal{M} , and the subdifferential $\partial f(x)$ is “continuous” at \bar{x} along the manifold.
2. For any point near $(\bar{x}, f(\bar{x}))$, there exists a unique nearest point to it in the *epigraph* $\{ (x, t) \mid t \geq f(x) \}$.
3. The subdifferential $\partial f(\bar{x})$ spans the normal space to the manifold, $N_{\mathcal{M}}(\bar{x})$.

We now see why these properties hold, using elementary arguments. Along the manifold \mathcal{M} , we have that $f(x) = \max_{i \in \bar{I}} f_i(x) = f_j(x)$ for any $j \in \bar{I}$. Therefore, the function f is smooth relative to \mathcal{M} .

The *generalized directional derivative* $f^\circ(x; d)$ is defined (for any Lipschitz function f) by

$$f^\circ(x; d) = \liminf_{y \rightarrow x, t \searrow 0} \frac{f(y + td) - f(y)}{t}.$$

A direct calculation shows that the generalized directional derivative for this example is, for $x \in \mathcal{M}$,

$$f^\circ(x; d) = \max_{i \in \bar{I}} \langle \nabla f_i(x), d \rangle,$$

which is continuous in x for all d . Since the function f is Lipschitz continuous, we have that

$$\partial f(x) = \{v \mid \langle v, d \rangle \leq f^\circ(x; d) \text{ for all } d\} \text{ for } x \in \mathcal{M}.$$

Hence, the subdifferential $\partial f(x)$ is continuous along the manifold \mathcal{M} .

For any (y, r) close to $(\bar{x}, f(\bar{x}))$, the following optimization problem can be

used to find the nearest point to (y, r) in epigraph of f :

$$\mathcal{P}(y, r) \begin{cases} \min_{(y,t) \in \mathbf{R}^n \times \mathbf{R}} |y - x|^2 + (r - t)^2 \\ \text{s.t. } f_i(x) \leq t, i \in \mathcal{I}. \end{cases}$$

First note that $(\bar{x}, f(\bar{x}))$ is the unique minimizer of $\mathcal{P}(\bar{x}, f(\bar{x}))$. It is not hard to check that $\mathcal{P}(\bar{x}, f(\bar{x}))$ satisfies the assumptions of Theorem 1.4. Therefore, for all (y, r) close to $(\bar{x}, f(\bar{x}))$, we have that $\mathcal{P}(y, r)$ has a unique solution. Then property 2 follows. It is easy to verify the last property from explicit representations of $\partial f(\bar{x})$ and $N_{\mathcal{M}}(\bar{x})$.

The active manifold \mathcal{M} captures the key nonsmooth features of the optimization problem under reasonable conditions. To be more precise, recall that we are assuming the “strict complementarity” condition from classical optimization (SCS). In the nonsmooth language, this amounts to a non-degeneracy condition holding at \bar{x} , specifically that 0 lies in the *relative interior* of the subdifferential $\partial f(\bar{x})$. Then the exact counterparts of a variety of theorems from nonlinear optimization can be rephrased in this setting in terms of the active manifold.

First-order necessary conditions. Since we have

$$\partial f(\bar{x}) = \left\{ \sum_{i \in \bar{\mathcal{I}}} \lambda_i \nabla f_i(\bar{x}) : \lambda_i \geq 0, \sum_{i \in \bar{\mathcal{I}}} \lambda_i = 1 \right\},$$

the KKT conditions (1.2) are equivalent to $0 \in \partial f(\bar{x})$. In the language of variational analysis this says that \bar{x} is *Clarke stationary*.

Second-order optimality conditions. The second-order sufficient conditions (1.5) require that the vector $(\bar{x}, \bar{\lambda})$ satisfy the KKT conditions and $w^T \sum_{i \in \bar{\mathcal{I}}} \bar{\lambda}_i \nabla^2 f_i(\bar{x}) w > 0$ for any nonzero w satisfying $\nabla f_i(\bar{x})^T w = 0$ for all $i \in \bar{\mathcal{I}}$. Notice that these critical directions w are the same as the tangent directions to the manifold \mathcal{M} . With the help of a little classical calculus,

we see therefore that the second-order conditions require that function f grows quadratically along the manifold \mathcal{M} . Therefore, the second-order optimality conditions can be interpreted as stating that f growing quadratically on the manifold around \bar{x} implies that f in fact grows quadratically everywhere around \bar{x} .

Active-set identification. The identification theorem 1.5 applied to problem (1.8) says that any point with the KKT conditions nearly satisfied must lie on the active manifold. More precisely, suppose that we have a sequence (x_n, y_n) with $y_n \in \partial f(x_n)$ converging to $(\bar{x}, 0)$. This condition guarantees that x_n nearly satisfies the KKT conditions for large n , which implies that x_n will eventually lie on the active manifold. In this way, the identification theorem 1.5 can be rewritten by using active manifold language: if $x_n \rightarrow \bar{x}$, $y_n \rightarrow 0$, $y_n \in \partial f(x_n)$, then $x_n \in \mathcal{M}$ for large n .

Sensitivity analysis. The manifold \mathcal{M} consists of all nearby approximately stationary points: for small $\delta > 0$

$$\mathcal{M} = (\partial f)^{-1}(\delta B) \text{ locally around } \bar{x},$$

where B is a unit ball.

To see this, we first prove $(\partial f)^{-1}(\delta B) \subset \mathcal{M}$ locally around \bar{x} . If not, there exists a sequence $(x_n, y_n) \rightarrow (\bar{x}, 0)$ with $y_n \in \partial f(x_n)$ and $x_n \notin \mathcal{M}$. This is contradictory to the identification result above.

For the other direction, it is sufficient to prove that for any sequence $x_n \in \mathcal{M} \rightarrow \bar{x}$ there exists a sequence $v_n \rightarrow 0$ such that $v_n \in \partial f(x_n)$ for large n , which can be derived by the fact that ∂f is continuous at \bar{x} relative to \mathcal{M} . Hence, we have $\mathcal{M} = (\partial f)^{-1}(\delta B)$ locally around \bar{x} .

This example illustrates how, by using the “active manifold” idea, the notion of

active sets can be captured in a more geometric manner and can be generalized to a broader setting.

1.3 Partial smoothness

The above example motivates the idea of “partial smoothness”, which was first introduced in [17]. This idea dates back to “identifiable surfaces” [46] and to “ \mathcal{UV} decomposition” [25]. It abstracts the geometry of the active set, and characterizes the essential smooth structure associated with “activity” in the nonsmooth setting. Partial smoothness not only captures a broad class of nonsmooth functions, but also enjoys a powerful calculus and sensitivity theory. All these nice properties make partial smoothness a promising candidate for understanding practical nonsmooth optimization.

The essential idea of partial smoothness is that nonsmoothness is associated with a manifold. Recall a manifold \mathcal{M} is, locally, a zero-set of finitely many smooth functions with linearly independent gradients. At any element, its normal space is the linear space spanned by the gradients of the functions, and its tangent space is the orthogonal complement space of its normal space. Given a manifold $\mathcal{M} \subset \mathbf{R}^n$, a partly smooth function f relative to the manifold, loosely speaking, behaves smoothly along the manifold and “sharply” along normal directions to the manifold, and furthermore the subdifferential of f is continuous along the manifold. For example, the function $f(x, y) = x^2 + |y|$ is partly smooth relative to the x -axis.

Partial smoothness generalizes the idea of active sets from classical nonlinear optimization, and also includes many other nonsmooth functions. For convex

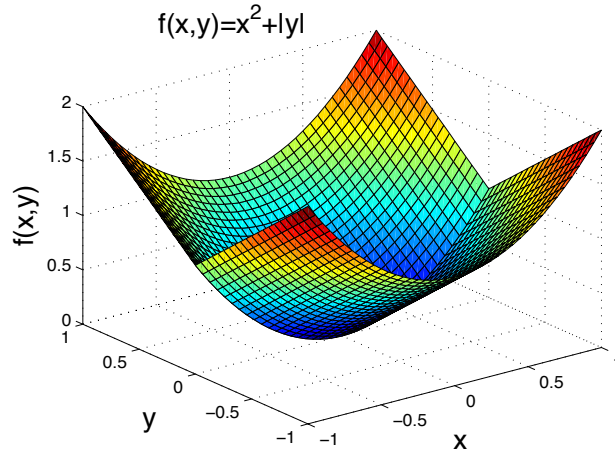


Figure 1.1: A partly smooth function with $\mathcal{M} = \{(x, 0)\}$

functions, the “ \mathcal{UV} decomposition” technique [26] is closely related to partial smoothness. Moreover, the partly smooth philosophy also applies in less classical nonsmooth optimization domains, such as eigenvalue optimization [44] and spectral abscissa minimization [4]. Furthermore, partial smoothness is mathematically elegant, having a powerful calculus and sensitivity theory [17]. As a result, partial smoothness can serve as a unifying language for studying practical nonsmooth optimization beyond classical nonlinear programming.

1.4 Tilt stability

For simplicity, we confine ourselves for the moment to unconstrained optimization. Chapter 3 of this dissertation concerns second-order optimality conditions and sensitivity analysis for nonsmooth optimization in the partly smooth setting. Our initial motivation stems from [38], which studies the behavior of a minimizing point when an objective function is tilted by a small linear term,

from the perspective of second-order conditions. The article [38] introduces the concept of tilt stability to characterize the case when the minimizing point depends Lipschitz continuously on the tilt perturbation. It starts from the observation that, for a C^2 smooth objective function, a local minimizer has this desirable property if and only if the corresponding Hessian matrix is positive definite. The authors of [38] then extend this result to nonsmooth objective functions using the “positivity” of the generalized Hessian mapping of Mordukhovich.

There are several complicated and sophisticated approaches to second-order variational analysis [43]. The generalized Hessian mapping is very natural and compelling, since it simply relies on the two sequential applications of the “normal cone” construction. The normal cone concept is very basic and fundamental in variational analysis. When the function is C^2 smooth, the generalized Hessian mapping is simply the classical Hessian matrix.

Despite computational difficulty, the generalized Hessian mapping is a fundamental tool in the study of nonsmooth optimization. In concrete optimization settings, what does the generalized Hessian mapping say, and how useful is it? The primary goal of Chapter 3 is to address this question in the partly smooth setting. Consider again, for a moment, the nonsmooth example 1.7. We studied tilt perturbations for that example and characterized tilt stability by using classical sensitivity theory. Remarkably, this case is typical for partly smooth functions. Under reasonable reasons, for partly smooth functions, generalized Hessians are easy to compute, and the tilt stability theorem [38, Thm 1.3] can be interpreted in a classical way. Note that Chapter 3 is based on the paper [22], accepted for publication in the *SIAM Journal on Optimization*.

In Chapter 5, we talk about further developments regarding tilt stability and

partial smoothness, in a discussion broken down into three parts.

1. Chapter 3 shows that computing the generalized Hessians in the partly smooth setting is relatively easy. We further develop several calculus rules in Chapter 5, which allow us to compute generalized Hessians for a variety of partly smooth functions.
2. The paper [5] derives second-order sufficient conditions for local minimizers of a specific interesting class of nonsmooth functions. We illustrate our results on this class of functions, recovering the same conditions.
3. The paper [16] characterizes the “full stability” of local solutions to finite-dimensional parameterized optimization problems, a condition for which the tilt stability case turns out to be a special case. Tilt stability concerns tilt perturbations, which correspond to adding a small linear term to the objective, while full stability not only concerns tilt perturbations, but also “basic” perturbations, which correspond to shifting the parameter values in an objective. Loosely speaking, a local solution being fully stable means the optimal solution varies in a Lipschitzian manner with respect to both basic and tilt perturbations. In Chapter 5, we will extend our results about tilt stability to full stability in the partly smooth setting.

1.5 Nonsmoothness and the BFGS method

As we discussed above, much of the motivation for partial smoothness comes from active-set methods. For example, to minimize the partly smooth function $\max_{i \in I} \{f_i(x)\}$, active-set methods ultimately search for solutions on the active manifold. As this example illustrates, active-set methods depend on the

geometric structure of the problems explicitly. Next we turn to a different algorithm whose execution seems to be independent of the explicit geometric structure. Nonetheless, the method typically seems to discover, implicitly, the partly smooth structure in the problems. This leads to another perspective of this thesis: study the behavior of the BFGS variable metric method on nonsmooth functions.

We study the behavior of the standard BFGS variable metric method for smooth minimization, when applied to nonsmooth functions. The theory for BFGS applied to convex smooth functions is well established: Powell [42] showed that BFGS with a Wolfe inexact line search converges to a minimizer, when applied to a twice-differentiable convex function with bounded level sets. However, there is no corresponding convergence result for nonconvex smooth functions. Although various authors [41, 24, 23] made some progress by restricting the function class or modifying the method, the convergence theory for the BFGS algorithm on nonconvex functions remains poorly understood. There is even less experience with the BFGS method on nonsmooth functions. The success of variable metric methods on nonsmooth functions was observed many years ago [15], but it seems very challenging to give any rigorous convergence analysis.

Recent work by Lewis and Overton [18] gives a detailed analysis of the BFGS method with an exact line search on one very particular example: the Euclidean norm function in \mathbf{R}^2 . While very special, the analysis illustrates how BFGS can work well on nonsmooth functions. A companion paper [19] investigates the behavior of BFGS with a suitable inexact line search on some nonsmooth examples: the authors observe that this inexact-line-search BFGS method typically

converges to Clarke stationary points, and they pose the following challenge, to prove or disprove.

Conjecture 1.1. *Consider any locally Lipschitz, semi-algebraic function f with bounded level sets, and choose the initial point x_0 and an initial inverse Hessian estimate H_0 randomly. With probability one, the BFGS method generates an infinite sequence of iterates, for which any cluster point \bar{x} is Clarke stationary, and furthermore the sequence of all function trial values converges to $f(\bar{x})$ R -linearly.*

For more precise details on the terminology, see [18, Challenge 7.1].

Chapter 4 is largely motivated by these two papers. We highlight further the success of line search BFGS method on some nonsmooth examples, and analyze the potential reasons. By way of contrast, we illustrate the potential bad behavior of the line-search BFGS method by constructing a nonsmooth function on which the method converges to a point at which there exist directions of linear descent. Our goal, throughout, is simply insight into the line-search BFGS method in the nonsmooth case.

As context, we also briefly discuss the behavior of a trust-region BFGS method when applied to nonsmooth functions. The line search and trust region philosophies for updating the current point of course differ considerably: trust region methods [6] approximate the original problem in a “trust region” by a quadratic subproblem, and take a corresponding step at each iteration. The trust-region BFGS method we discuss for illustration is a simple combination of the trust region method in [32] with the BFGS algorithm in [19]. Our purpose is to understand the fundamental difference between these two different strategies in nonsmooth optimization. Chapter 4 is based on the submitted manuscript [21].

CHAPTER 2

NOTATION AND DEFINITIONS

2.1 Introduction

In this chapter, we introduce the common notation, definitions and results used throughout this dissertation. Unless other stated, we follow the notation and terminology of [43].

2.2 Basics

We only consider finite-dimensional spaces \mathbf{R}^n in this dissertation. We denote the extended reals by $\overline{\mathbf{R}} = [-\infty, +\infty]$. Given a set $S \subset \mathbf{R}^n$, its *relative interior* (when S is convex) is denoted by $\text{ri } S$, and its *indicator function* δ_S is defined by

$$\delta_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S. \end{cases}$$

Definition 2.1. A function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is proper if $f(x) < \infty$ for at least one $x \in \mathbf{R}^n$, and $f(x) > -\infty$ for all $x \in \mathbf{R}^n$. Its domain and epigraph are defined to be

$$\text{dom } f = \{ x \mid f(x) < \infty \},$$

and

$$\text{epi } f = \{ (x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x) \}.$$

Definition 2.2. A function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is lower semicontinuous on \mathbf{R}^n if its epigraph is closed. A function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is locally lower semicontinuous at x if its epigraph is locally closed at $(x, f(x))$.

Definition 2.3. Given a set-valued mapping $F : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, its graph, domain, and inverse mapping F^{-1} are defined by

$$\text{gph } F = \{ (x, y) : y \in F(x) \},$$

$$\text{dom } F = \{ x : F(x) \neq \emptyset \},$$

and

$$x \in F^{-1}(y) \Leftrightarrow y \in F(x).$$

2.3 Variational analysis

Subgradients are a fundamental tool in variational analysis, and there are several versions of them. In this dissertation, we use the following subgradients.

Definition 2.4. Consider a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ and a point \bar{x} with $f(\bar{x})$ finite. For a vector $v \in \mathbf{R}^n$, one says:

1. v is a regular subgradient of f at \bar{x} , written $v \in \hat{\partial}f(\bar{x})$, if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(|x - \bar{x}|);$$

and, the regular subdifferential of f at \bar{x} is defined as the set $\hat{\partial}f(\bar{x})$;

2. v is a (general) subgradient of f at \bar{x} , written $v \in \partial f(\bar{x})$, if there are sequences $x^v \rightarrow \bar{x}$ and $v^v \in \hat{\partial}f(x^v)$ with $f(x^v) \rightarrow f(\bar{x})$ and $v^v \rightarrow v$; furthermore the subdifferential of f at \bar{x} is defined as the set $\partial f(\bar{x})$;

3. v is a horizon subgradient of f at \bar{x} , written $v \in \partial^\infty f(\bar{x})$, if the same holds as in condition 2, except that instead of $v^v \rightarrow v$ one has $\lambda^v v^v \rightarrow v$ for some sequence $\lambda^v \searrow 0$; and, the horizon subdifferential of f at \bar{x} is defined as the set $\partial^\infty f(\bar{x})$.

Definition 2.5. Given a set $S \subset \mathbf{R}^n$, its regular normal cone at a point \bar{x} is defined by $\hat{N}_S(\bar{x}) := \hat{\partial}\delta_S(\bar{x})$, and its normal cone at x is defined by $N_S(\bar{x}) := \partial\delta_S(\bar{x})$.

Definition 2.6. For a set $C \subset \mathbf{R}^n$, the horizon cone is defined by

$$C^\infty = \begin{cases} \{ x \mid \exists x_n \in C, \lambda_n \searrow 0, \lambda_n x_n \rightarrow x \} & C \neq \emptyset \\ \{ 0 \} & C = \emptyset. \end{cases}$$

Definition 2.7. A set $S \subset \mathbf{R}^n$ is regular at a point $\bar{x} \in S$ if it is locally closed at \bar{x} and $N_S(\bar{x}) = \hat{N}_S(\bar{x})$. A function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is regular at \bar{x} if $\text{epi } f$ is regular at $(\bar{x}, f(\bar{x}))$.

There are strong connections between subgradients and the geometry of epigraphs.

Theorem 2.1. [43, Thm 8.9] For function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ and any point \bar{x} at which f is finite, one has

$$\begin{aligned} \hat{\partial}f(\bar{x}) &= \left\{ v \mid (v, -1) \in \hat{N}_{\text{epi } f}(\bar{x}, f(\bar{x})) \right\}, \\ \partial f(\bar{x}) &= \left\{ v \mid (v, -1) \in N_{\text{epi } f}(\bar{x}, f(\bar{x})) \right\}, \\ \partial^\infty f(\bar{x}) &\subset \left\{ v \mid (v, 0) \in N_{\text{epi } f}(\bar{x}, f(\bar{x})) \right\}. \end{aligned}$$

The last relationship holds with equality when f is locally lower semicontinuous at \bar{x} , and then

$$N_{\text{epi } f}(\bar{x}, f(\bar{x})) = \left\{ \lambda(v, -1) \mid v \in \partial f(\bar{x}), \lambda > 0 \right\} \cup \left\{ (v, 0) \mid v \in \partial^\infty f(\bar{x}) \right\}.$$

On the other hand, whenever $\hat{\partial}f(\bar{x}) \neq \emptyset$ one has

$$\hat{N}_{\text{epi } f}(\bar{x}, f(\bar{x})) = \left\{ \lambda(v, -1) \mid v \in \hat{\partial}f(\bar{x}), \lambda > 0 \right\} \cup \left\{ (v, 0) \mid v \in \hat{\partial}f(\bar{x})^\infty \right\}.$$

The following corollary gives an alternative characterization for a function being regular.

Corollary 2.1. [43, Cor.8.11] For a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ and a point \bar{x} with $f(\bar{x})$ finite and $\partial f(\bar{x}) \neq \emptyset$, one has f regular at \bar{x} if and only if f is locally lower-semicontinuous at \bar{x} with

$$\partial f(\bar{x}) = \hat{\partial} f(\bar{x}), \quad \partial^\infty f(\bar{x}) = (\hat{\partial} f(\bar{x}))^\infty.$$

Subdifferential calculus allows us to compute and determine subgradients efficiently.

Theorem 2.2. [43, Thm 10.6] Suppose $f(x) = g(F(x))$ for a proper, lower semicontinuous function $g : \mathbf{R}^m \rightarrow \overline{\mathbf{R}}$ and a smooth mapping $F : \mathbf{R}^n \rightarrow \mathbf{R}^m$. Then at any point $\bar{x} \in \text{dom } f = F^{-1}(\text{dom } g)$ one has

$$\hat{\partial} f(\bar{x}) \supset \nabla F(\bar{x})^* \hat{\partial} g(F(\bar{x})).$$

If the only vector $y \in \partial^\infty g(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = 0$ is $y = 0$, then one also has

$$\partial f(\bar{x}) \subset \nabla F(\bar{x})^* \partial g(F(\bar{x})), \quad \partial^\infty f(\bar{x}) \subset \nabla F(\bar{x})^* \partial^\infty g(F(\bar{x})).$$

If in addition g is regular at $F(\bar{x})$, then f is regular at \bar{x} and

$$\partial f(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x})), \quad \partial^\infty f(\bar{x}) = \nabla F(\bar{x})^* \partial^\infty g(F(\bar{x})).$$

Theorem 2.3. [43, Cor 10.9] Suppose function $f = f_1 + \dots + f_m$ for proper, lower semicontinuous functions $f_i : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$, and let $\bar{x} \in \text{dom } f$. Then

$$\hat{\partial} f(\bar{x}) \supset \hat{\partial} f_1(\bar{x}) + \dots + \hat{\partial} f_m(\bar{x}).$$

Under the condition that the only combination of vectors $v_i \in \partial^\infty f_i(\bar{x})$ with $v_1 + \dots + v_m = 0$ is $v_1 = v_2 = \dots = v_m = 0$, one also has that

$$\partial f(\bar{x}) \subset \partial f_1(\bar{x}) + \dots + \partial f_m(\bar{x}),$$

$$\partial^\infty f(\bar{x}) \subset \partial^\infty f_1(\bar{x}) + \dots + \partial^\infty f_m(\bar{x}).$$

If also each f_i is regular at \bar{x} . then f is regular at \bar{x} and

$$\partial f(\bar{x}) = \partial f_1(\bar{x}) + \cdots + \partial f_m(\bar{x}),$$

$$\partial^\infty f(\bar{x}) = \partial^\infty f_1(\bar{x}) + \cdots + \partial^\infty f_m(\bar{x}).$$

CHAPTER 3
PARTIAL SMOOTHNESS, TILT STABILITY, AND GENERALIZED
HESSIANS

3.1 Introduction

The distinction between active and inactive constraints is fundamental throughout optimization, underlying optimality conditions, sensitivity analysis, and algorithm design. The notion of “partial smoothness” [17] (along with analogues such as “identifiable surfaces” [46] and “ \mathcal{UV} decompositions” [25]) captures the essential geometry associated with activity, and in a fashion suitable for generalization beyond classical nonlinear programming into such domains as semidefinite programming. Partial smoothness illustrates well the power of modern variational analysis as a unifying language for concrete optimization. It is, furthermore, a generic property in concrete settings such as semialgebraic convex optimization [2].

The partly smooth setting allows intuitive and appealing statements of second-order optimality conditions and associated sensitivity analysis around a “nondegenerate” critical point (where the subdifferential contains zero in its relative interior) [17, 12]. In this case the second-order conditions boil down to the classical smooth case, resulting in the idea of a “strong critical point”. Much more general second-order variational analysis is available: see, for example, the monographs [43, 3, 27]. A particularly attractive approach is via Mordukhovich’s generalized Hessian [27]. That particular theoretical development is natural and compelling, relying simply on two sequential applications of the normal cone construction basic to variational analysis, but computing the

generalized Hessian in general can be hard.

Despite computational challenges, the generalized Hessian is clearly a fundamental tool. In particular, [38] considers one of the most basic questions of sensitivity analysis: under what conditions does a local minimizer of a function depend in a Lipschitz fashion on linear perturbations to the function? Assuming that the function is both “prox-regular” and “subdifferentially continuous” (as holds, for example, for a composition of a continuous convex function with a C^2 -smooth map), this “tilt stability” property turns out to be equivalent to positive-definiteness of the generalized Hessian [38].

We prove two main results. We first show that, for partly smooth, prox-regular, subdifferentially continuous functions, the generalized Hessian is easy to compute at a nondegenerate critical point. Then, as a simple consequence using the characterization of [38], we show that, in this setting, strong criticality is actually equivalent to tilt stability.

3.2 Generalized Hessian mappings of simple nonsmooth Functions

Unless otherwise stated, we follow the notation and terminology of [43]. In particular, $\overline{\mathbf{R}}$ denotes the extended reals, $\partial f(x)$ denotes the set of subgradients of a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ at a point $x \in \mathbf{R}^n$, and $N_S(x)$ denotes the normal cone to set $S \subset \mathbf{R}^n$ at a point $x \in \mathbf{R}^n$. We denote the graph of a set-valued mapping F by $\text{gph } F$.

The concept of tilt stability, introduced in [38], characterizes the case where

adding a small linear term to a function shifts the minimizing point in a Lipschitzian manner and where that point is locally unique.

Definition 3.1. A point \bar{x} will be said to give a tilt stable local minimum of the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ if $f(\bar{x})$ is finite and there exists a $\delta > 0$ such that the mapping

$$M : v \mapsto \operatorname{argmin}_{|x-\bar{x}| \leq \delta} \{ f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle \},$$

is single-valued and Lipschitzian on some neighborhood of $v = 0$, with $M(0) = \bar{x}$.

For a C^2 -smooth function f with $\nabla f(\bar{x}) = 0$, the point \bar{x} gives a tilt stable local minimum of f if and only if $\nabla^2 f(\bar{x})$ is positive definite, according to [38, Prop. 1.2]. This fact has been extended to nonsmooth functions in terms of the positivity of a certain generalized Hessian mapping [38].

Definition 3.2. For any point \bar{x} and any subgradient $\bar{v} \in \partial f(\bar{x})$, define the generalized Hessian mapping $\partial^2 f(\bar{x}|\bar{v}) : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ by

$$\partial^2 f(\bar{x}|\bar{v}) : w \mapsto \left\{ z \mid (z, -w) \in N_{\operatorname{gph} \partial f}(\bar{x}, \bar{v}) \right\}.$$

For a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ having $0 \in \partial f(\bar{x})$, [38, Thm. 1.3] shows that, under certain assumptions, the point \bar{x} gives a tilt stable local minimum of f if and only if $0 \in \partial f(\bar{x})$ and the mapping $\partial^2 f(\bar{x}|0)$ is positive definite in the sense that

$$\langle z, w \rangle > 0 \text{ whenever } z \in \partial^2 f(\bar{x}|0)(w), \ w \neq 0.$$

To compute the generalized Hessian mapping, it is sufficient to know $N_{\operatorname{gph} \partial f}$. Let's introduce the definition of a manifold first.

Definition 3.3. We say that a set $\mathcal{M} \subset \mathbf{R}^n$ is a C^k -smooth manifold ($k = 1, 2$) of codimension m around a point $\bar{x} \in \mathbf{R}^n$ if $\bar{x} \in \mathcal{M}$ and there is an open set $V \subset \mathbf{R}^n$ such that

$$\mathcal{M} \cap V = \{ x \in V \mid \Phi_i(x) = 0, \ i = 1, \dots, m \}$$

where Φ_i are C^k -smooth functions with $\nabla\Phi_i(\bar{x})$ linearly independent.

In this case, it is well known that the tangent space to \mathcal{M} at \bar{x} is given by

$$T_{\mathcal{M}}(\bar{x}) = \{\nabla\Phi_i(\bar{x})\}^\perp$$

and the normal space to \mathcal{M} at \bar{x} is

$$N_{\mathcal{M}}(\bar{x}) = \left\{ \sum_i \lambda_i \nabla\Phi_i(\bar{x}) \mid \lambda \in \mathbf{R}^m \right\}.$$

We call $\Phi_i(x) = 0$ local equations for \mathcal{M} .

Our immediate aim is to compute the normal cone to the graph of the normal cone mapping $N_{\mathcal{M}}$. An explicit formula follows from [13, Thm. 3.1], [35, Thm. 7], and [30, Thm. 3.1]- see also [29, Thm. 3.4] and [28, Thm. 1.127]. Here, for completeness and to fix our later notation, we give a self-contained classical approach.

Definition 3.4. When $F : U \rightarrow \mathbf{R}^m$ is a C^1 -smooth mapping of an open set $U \subset \mathbf{R}^n$, the rank of F at a point $x \in U$ is defined as the dimension of the range of the gradient $\nabla F(x)$.

The next result shows that functions of constant rank locally have simple structure.

Theorem 3.1 (constant rank). Suppose $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$ are open sets and $F : U \rightarrow V$ is a smooth map with constant rank k . For any point $p \in U$, there exist open sets $U_0 \subset U$ containing p , $V_0 \subset V$ containing $F(p)$ and diffeomorphisms $\varphi : U_0 \rightarrow \varphi(U_0)$, $\psi : V_0 \rightarrow \psi(V_0)$, with $F(U_0) \subset V_0$, such that

$$\psi \circ F \circ \varphi^{-1}(x_1, \dots, x_k, x_{k+1}, \dots, x_m) = (x_1, \dots, x_k, 0, \dots, 0).$$

Proof. See [14, Thm. 7.8]. □

Note: The above theorem is also true for C^k -smooth functions ($k \geq 1$), in which case φ and ψ are C^k diffeomorphisms. The following is standard, but we include a proof for convenience.

Proposition 3.1 (Immersion). *If \mathcal{M} is a C^2 -smooth manifold of codimension m around a point \bar{x} , then there exist an open set $U \subset \mathbf{R}^{n-m}$ and an injective C^2 -smooth mapping $G : U \rightarrow \mathbf{R}^n$ with $G(U) = \mathcal{M}$ locally around \bar{x} .*

Proof. Since \mathcal{M} is a C^2 -manifold of codimension m around \bar{x} , then there exists an open set $V \subset \mathbf{R}^n$ such that

$$\mathcal{M} \cap V = \{ x \in V \mid \Phi_i(x) = 0, i = 1, \dots, m \},$$

where Φ_i are C^2 -smooth with $\nabla \Phi_i(\bar{x})$ linearly independent. Shrinking V if necessary, we can assume that $\nabla \Phi_i(x)$ are linearly independent for all $x \in V$. The implicit function theorem is stated as follows: Let $F : \mathbf{R}^{n+m} \rightarrow \mathbf{R}^m$ be a continuously differentiable function, and let \mathbf{R}^{n+m} have coordinates (x, y) . Fix a point $(a_1, \dots, a_n, b_1, \dots, b_m) = (a, b)$ with $F(a, b) = c$, where $c \in \mathbf{R}^m$. If the matrix $[(\partial F_i / \partial y_j)(a, b)]$ is invertible, then there exists an open set U containing a , an open set V containing b , and a unique continuously differentiable function $g : U \rightarrow V$ such that

$$\{ (x, g(x)) \mid x \in U \} = \{ (x, y) \in U \times V \mid F(x, y) = c \}.$$

According to the implicit function theorem, without loss of generality there exist open sets $U \subset \mathbf{R}^{n-m}$, $W \subset \mathbf{R}^m$ and a C^2 -smooth function $g : U \rightarrow W$ with $\bar{x} \in U \times W \subset V$ such that

$$\{ (u, g(u)) \in U \times W \} = \{ (u, w) \in U \times W \mid \Phi_i(u, w) = 0, i = 1, \dots, m \}.$$

Then define an injective function $G : U \rightarrow \mathbf{R}^n$ by

$$G(u) = (u, g(u)).$$

It is easy to check that $G(U) = \mathcal{M}$ locally around \bar{x} . □

Proposition 3.2 (tangents to immersions). *Let $U \subset \mathbf{R}^m$ be an open set with a point $\bar{u} \in U$, and $G : U \rightarrow \mathbf{R}^n$ be C^k -smooth with $\nabla G(\bar{u})$ full rank. Then there exists an open set $U_0 \subset U$ containing \bar{u} such that $G(U_0)$ is a C^k -manifold around $G(\bar{u})$ and $T_{G(U_0)}(G(u)) = R(\nabla G(u))$ for all $u \in U_0$.*

Proof. Since $G : U \rightarrow \mathbf{R}^n$ is C^k -smooth with $\nabla G(\bar{u})$ full rank, then G is of constant rank m around \bar{u} . According to Theorem 3.1, there exist open sets $U_0 \subset \mathbf{R}^m$ containing \bar{u} , $V_0 \subset \mathbf{R}^n$ containing $G(\bar{u})$, and diffeomorphisms $\varphi : U_0 \rightarrow \varphi(U_0)$, $\psi : V_0 \rightarrow \psi(V_0)$, with $U_0 \subset U$ and $G(U_0) \subset V_0$, such that

$$\psi \circ G \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0).$$

Hence,

$$G(U_0) \cap V_0 = \{x \in V_0 : \psi_i(x) = 0, i = m+1, \dots, n\},$$

where $\nabla \psi_i(x)$ are linearly independent on V_0 . Therefore $G(U_0)$ is a manifold around $G(\bar{u})$. Hence

$$T_{G(U_0)}(G(u)) = \text{Ker}(\nabla \Phi(G(u))),$$

where $\Phi(x) = (\psi_{m+1}, \dots, \psi_n)$. Since $\Phi \circ G(u) = 0$ for any $u \in U_0$, then by the chain rule we get

$$\nabla \Phi(G(u)) \nabla G(u) = 0.$$

Therefore $R(\nabla G(u)) \subset \text{Ker}(\nabla \Phi(G(u)))$. Since $\dim(\nabla G(u)) = \dim(\text{Ker}(\nabla \Phi(G(u)))) = m$, then

$$T_{G(U_0)}(G(u)) = \text{Ker}(\nabla \Phi(G(u))) = R(\nabla G(u)).$$

□

Theorem 3.2 (normals to the normal bundle). *Suppose a point $\bar{x} \in V \subset \mathbf{R}^n$, where V is an open set, and that $\Phi_i : V \rightarrow \mathbf{R}$ ($i = 1, \dots, m$) are C^2 -smooth functions with $\nabla\Phi_i(\bar{x})$ linearly independent. Then there exists an open set $V' \subset V$ containing \bar{x} such that*

$$\mathcal{M} = \{ x \in V' \mid \Phi_i(x) = 0, i = 1, \dots, m \}$$

is a C^2 -smooth manifold around \bar{x} with

$$T_{\mathcal{M}}(x) = \{\nabla\Phi_i(x)\}^\perp \text{ and } N_{\mathcal{M}}(x) = \left\{ \sum_i \lambda_i \nabla\Phi_i(x) \mid \lambda \in \mathbf{R}^m \right\} \quad (3.1)$$

for any $x \in \mathcal{M}$. Furthermore, the normal bundle $\text{gph } N_{\mathcal{M}}$ is a C^1 -smooth manifold around $(x, \sum \lambda_i \nabla\Phi_i(x))$ and

$$N_{\text{gph } N_{\mathcal{M}}} \left(x, \sum_i \lambda_i \nabla\Phi_i(x) \right) = \left\{ (z, w) \mid w \in T_{\mathcal{M}}(x), z + \sum_i \lambda_i \nabla^2\Phi_i(x)w \in N_{\mathcal{M}}(x) \right\}$$

for any $x \in \mathcal{M}$ and $\lambda \in \mathbf{R}^m$.

Proof. Since \mathcal{M} is a C^2 -smooth manifold of codimension m , we can choose $G : U \rightarrow \mathbf{R}^n$ with $G(\bar{u}) = \bar{x}$ as in Proposition 3.1. According to the proof of Proposition 3.1, it is easy to deduce that $\nabla G(u)$ is full rank for any $u \in U$. Moreover, (3.1) holds. Define the following C^1 -smooth function $F : U \times \mathbf{R}^m \rightarrow \mathbf{R}^n \times \mathbf{R}^n$ by

$$F(u, \lambda) = \left(G(u), \sum \lambda_i \nabla\Phi_i(G(u)) \right), \text{ where } u \in U, \lambda \in \mathbf{R}^m.$$

Let's compute $T_{\text{gph } N_{\mathcal{M}}}(x, \sum_i \lambda_i \nabla\Phi_i(x))$ first. Since

$$\nabla F(u, \lambda) = \begin{pmatrix} \nabla G(u) & 0 & \dots & 0 \\ \sum \lambda_i \nabla^2\Phi_i(G(u)) \nabla G(u) & \nabla\Phi_1(G(u)) & \dots & \nabla\Phi_m(G(u)) \end{pmatrix}$$

has full rank for any $(u, \lambda) \in U \times \mathbf{R}^m$, there exists an open set $U_0 \times W_0 \subset U \times \mathbf{R}^m$ such that locally around $(x, \sum_i \lambda_i \nabla\Phi_i(x))$ the set $F(U_0 \times W_0) = \text{gph } N_{\mathcal{M}}$ is a C^1 -smooth

manifold by Proposition 3.2. Moreover, we have that

$$\begin{aligned} T_{\text{gph } N_{\mathcal{M}}} \left(x, \sum_i \lambda_i \nabla \Phi_i(x) \right) &= R(\nabla F(u, \lambda)) \\ &= \left\{ \left(\nabla G(u)a, \sum_i \lambda_i \nabla^2 \Phi_i(G(u)) \nabla G(u)a + \sum_i b_i \nabla \Phi_i(G(u)) \right) \mid a \in \mathbf{R}^{n-m}, b \in \mathbf{R}^m \right\}. \end{aligned}$$

Then letting $z := \nabla G(u)a$ and $w := \sum_i \lambda_i \nabla^2 \Phi_i(G(u)) \nabla G(u)a + \sum_i b_i \nabla \Phi_i(G(u))$, we have

$$T_{\text{gph } N_{\mathcal{M}}} \left(x, \sum_i \lambda_i \nabla \Phi_i(x) \right) = \left\{ (z, w) \mid z \in T_{\mathcal{M}}(x), w - \sum_i \lambda_i \nabla^2 \Phi_i(x) z \in N_{\mathcal{M}}(x) \right\}.$$

Since $\text{gph } N_{\mathcal{M}}$ is a C^1 -smooth manifold around $(x, \sum_i \lambda_i \nabla \Phi_i(x))$, then

$N_{\text{gph } N_{\mathcal{M}}}(x, \sum_i \lambda_i \nabla \Phi_i(x)) = T_{\text{gph } N_{\mathcal{M}}}(x, \sum_i \lambda_i \nabla \Phi_i(x))^\perp$. We can calculate this set from the fact that for any linear map A and a linear subspace S

$$\{x \mid Ax \in S\}^\perp = A^* S^\perp.$$

In this case,

$$A = \begin{pmatrix} I & 0 \\ -\sum_i \lambda_i \nabla^2 \Phi_i(x) & I \end{pmatrix} \text{ and } S = \{ (u, v) \mid u \in T_{\mathcal{M}}(x), v \in N_{\mathcal{M}}(x) \}.$$

Therefore

$$N_{\text{gph } N_{\mathcal{M}}} \left(x, \sum_i \lambda_i \nabla \Phi_i(x) \right) = \left\{ (z, w) \mid w \in T_{\mathcal{M}}(x), z + \sum_i \lambda_i \nabla^2 \Phi_i(x) w \in N_{\mathcal{M}}(x) \right\}.$$

□

Note: The classic definition of a manifold is via “coordinate charts.” Then the manifold $\mathcal{M} \subset \mathbf{R}^n$ defined by Definition 3.3 can be identified as an embedded submanifold of \mathbf{R}^n according to [14, Prop. 8.12]. In this setting, Proposition 3.1 and 3.2 are standard results in smooth manifold theory.

Corollary 3.1 (generalized Hessians: smooth case). *Consider a point $\bar{x} \in V \subset \mathbf{R}^n$ where V is an open set and $\Phi_i : V \rightarrow \mathbf{R}$ ($i = 1, \dots, m$) are C^2 -smooth with $\nabla \Phi_i(\bar{x})$*

linearly independent. Then there exists an open set $V' \subset V$ containing \bar{x} such that

$$\mathcal{M} = \{ x \in V' \mid \Phi_i(x) = 0, i = 1, \dots, m \}$$

is a C^2 -smooth manifold around \bar{x} with the following property. Suppose $h : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is a C^2 -smooth function around \bar{x} with $0 \in \partial(h + \delta_{\mathcal{M}})(\bar{x})$. Then there exists a unique $\bar{\lambda} \in \mathbf{R}^m$ such that the Lagrangian $L = h + \sum_i \bar{\lambda}_i \Phi_i$ satisfies $\nabla L(\bar{x}) = 0$ and

$$\partial^2(h + \delta_{\mathcal{M}})(\bar{x}|0)(w) = \begin{cases} \nabla^2 L(\bar{x})w + N_{\mathcal{M}}(\bar{x}), & w \in T_{\mathcal{M}}(\bar{x}), \\ \emptyset, & w \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

Proof. Since $0 \in \partial(h + \delta_{\mathcal{M}})(\bar{x})$ and $\nabla \Phi_i(\bar{x})$ are linearly independent, there exists a unique $\bar{\lambda} \in \mathbf{R}^m$ such that $-\nabla h(\bar{x}) = \sum_i \bar{\lambda}_i \nabla \Phi_i(\bar{x})$. According to [38, Prop. 4.1], we have that for any $\bar{x} \in \mathcal{M}$ and $w \in \mathbf{R}^n$

$$\partial^2(h + \delta_{\mathcal{M}})(\bar{x}|0)(w) = \nabla^2 h(\bar{x})w + \partial^2 \delta_{\mathcal{M}}(\bar{x} | -\nabla h(\bar{x}))(w).$$

Since

$$\partial^2 \delta_{\mathcal{M}}(\bar{x} | -\nabla h(\bar{x})) : w \mapsto \left\{ z \mid (z, -w) \in N_{\text{gph } N_{\mathcal{M}}}(\bar{x}, -\nabla h(\bar{x})) \right\},$$

then this problem boils down to computing the normal cone of $\text{gph } N_{\mathcal{M}}$ at $(\bar{x}, -\nabla h(\bar{x}))$. According to Proposition 3.2, we have that for any $w \in T_{\mathcal{M}}(\bar{x})$

$$\partial^2 \delta_{\mathcal{M}}(\bar{x} | -\nabla h(\bar{x}))(w) = \partial^2 \delta_{\mathcal{M}} \left(\bar{x} \mid \sum_i \bar{\lambda}_i \nabla \Phi_i(\bar{x}) \right)(w) = \sum_i \bar{\lambda}_i \nabla^2 \Phi_i(\bar{x})w + N_{\mathcal{M}}(\bar{x}).$$

Hence

$$\begin{aligned} \partial^2(h + \delta_{\mathcal{M}})(\bar{x}|0)(w) &= \nabla^2 h(\bar{x})w + \partial^2 \delta_{\mathcal{M}}(\bar{x} | -\nabla h(\bar{x}))(w) \\ &= \begin{cases} \nabla^2 L(\bar{x})w + N_{\mathcal{M}}(\bar{x}), & w \in T_{\mathcal{M}}(\bar{x}), \\ \emptyset, & w \notin T_{\mathcal{M}}(\bar{x}). \end{cases} \end{aligned}$$

□

Since $N_{\text{gph } N_{\mathcal{M}}}$ is determined only by the geometry of \mathcal{M} , we can use intrinsic geometric objects to formulate it. Next, we will introduce the concept of covariant derivative and Hessian.

Definition 3.5. Let a C^2 -smooth manifold $\mathcal{M} \subset \mathbf{R}^n$ contain a point \bar{x} . We say a function $f : \mathcal{M} \rightarrow \mathbf{R}$ is C^2 -smooth around \bar{x} if there exists a representative function $h : \mathbf{R}^n \rightarrow \mathbf{R}$ which is C^2 -smooth around \bar{x} with $h|_{\mathcal{M}} = f|_{\mathcal{M}}$ locally around \bar{x} .

Let \mathcal{M} be a C^2 -smooth manifold around \bar{x} . Then the projection mapping $u \mapsto P_{\mathcal{M}}(\bar{x} + u)$ is well-defined and C^2 -smooth around 0 on $T_{\mathcal{M}}(\bar{x})$, as proved in [26].

Definition 3.6. Suppose $\mathcal{M} \subset \mathbf{R}^n$ is a C^2 -smooth manifold around a point \bar{x} and a function $f : \mathcal{M} \rightarrow \mathbf{R}$ is C^2 -smooth around \bar{x} . Then the covariant derivative $\nabla f_{\mathcal{M}}(\bar{x}) \in T_{\mathcal{M}}(\bar{x})$ is defined by

$$\langle \nabla_{\mathcal{M}} f(\bar{x}), u \rangle = \frac{d}{dt} f(P_{\mathcal{M}}(\bar{x} + tu))|_{t=0} \text{ for all } u \in T_{\mathcal{M}}(\bar{x}),$$

and the covariant Hessian $\nabla_{\mathcal{M}}^2 f(\bar{x}) : T_{\mathcal{M}}(\bar{x}) \times T_{\mathcal{M}}(\bar{x}) \rightarrow \mathbf{R}$ is the unique self-adjoint and bilinear map satisfying

$$\langle \nabla_{\mathcal{M}}^2 f(\bar{x})u, u \rangle = \frac{d^2}{dt^2} f(P_{\mathcal{M}}(\bar{x} + tu))|_{t=0} \text{ for all } u \in T_{\mathcal{M}}(\bar{x}).$$

This definition agrees with the classic definition of covariant derivative and Hessian using geodesics as proved in [26]. Suppose the function $f : \mathcal{M} \rightarrow \mathbf{R}$ is C^2 -smooth around \bar{x} . Let h be any C^2 -smooth representative of f around \bar{x} , and let C^2 -smooth functions Φ_i define local equations for \mathcal{M} . If $\nabla h(\bar{x}) \in N_{\mathcal{M}}(\bar{x})$, then using the notation of Corollary 3.1, there exists a unique $\bar{\lambda}$ such that $\nabla h(\bar{x}) + \sum \bar{\lambda}_i \nabla \Phi_i(\bar{x}) = 0$. Furthermore, the following results have been shown in [26]:

$$\nabla_{\mathcal{M}} f(\bar{x}) = P_{T_{\mathcal{M}}(\bar{x})} \nabla h(\bar{x}) \text{ and } \nabla_{\mathcal{M}}^2 f(\bar{x}) = P_{T_{\mathcal{M}}(\bar{x})} \nabla^2 L(\bar{x}) P_{T_{\mathcal{M}}(\bar{x})}.$$

Theorem 3.3 (generalized and covariant Hessians). *Suppose $\mathcal{M} \subset \mathbf{R}^n$ is a C^2 -smooth manifold around a point \bar{x} and the function $f : \mathcal{M} \rightarrow \mathbf{R}$ is C^2 -smooth around \bar{x} . Define the function $\tilde{f} : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ by*

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \mathcal{M}, \\ +\infty, & x \notin \mathcal{M}. \end{cases}$$

Then

$$0 \in \partial \tilde{f}(\bar{x}) \Leftrightarrow \nabla f_{\mathcal{M}}(\bar{x}) = 0,$$

and in that case

$$\partial^2 \tilde{f}(\bar{x}|0)(w) = \begin{cases} \nabla_{\mathcal{M}}^2 f(\bar{x})w + N_{\mathcal{M}}(\bar{x}) & w \in T_{\mathcal{M}}(\bar{x}) \\ \emptyset & w \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

Proof. Let h be a C^2 -smooth representative of f around \bar{x} . Then we have

$$\nabla_{\mathcal{M}} f(\bar{x}) = 0 \Leftrightarrow \nabla h(\bar{x}) \in N_{\mathcal{M}}(\bar{x}) \Leftrightarrow 0 \in \partial \tilde{f}(\bar{x}).$$

Let $\bar{\lambda}$ be the unique multiplier satisfying $\nabla h(\bar{x}) + \sum_i \bar{\lambda}_i \nabla \Phi_i(\bar{x}) = 0$. Since $\tilde{f}(x) = h(x) + \delta_{\mathcal{M}}(x)$, then for any $w \in T_{\mathcal{M}}(\bar{x})$ we have, by Corollary 3.1,

$$\begin{aligned} \partial^2 \tilde{f}(\bar{x}|0)(w) &= \partial^2 (h + \delta_{\mathcal{M}})(\bar{x}|0)(w) \\ &= \nabla^2 h(\bar{x})w + \sum_i \bar{\lambda}_i \nabla^2 \Phi_i(\bar{x})w + N_{\mathcal{M}}(\bar{x}) \\ &= P_{T_{\mathcal{M}}(\bar{x})} \left(\nabla^2 h(\bar{x}) + \sum_i \bar{\lambda}_i \nabla^2 \Phi_i(\bar{x}) \right) P_{T_{\mathcal{M}}(\bar{x})} w + N_{\mathcal{M}}(\bar{x}) \\ &= \nabla_{\mathcal{M}}^2 f(\bar{x})w + N_{\mathcal{M}}(\bar{x}). \end{aligned}$$

The result follows. □

We refer to functions of the form \tilde{f} as *extended- C^2 -smooth* at \bar{x} . The above theorem gives us some indication of how to calculate a generalized Hessian

mapping. The smooth manifold \mathcal{M} simplifies the calculation. “Partial smoothness”, which was introduced in [17], gives some underlying smooth structure for a nonsmooth function. In this chapter, we are going to show that for a partly smooth function relative to manifold \mathcal{M} , the local geometry of $\text{gph } \partial f(x)$ is determined by the restriction of f to \mathcal{M} , under certain assumptions. In this way, we can extend Theorem 3.1 to partly smooth functions.

3.3 Definitions and results

Definition 3.7. Suppose that $C \subset \mathbf{R}^n$ is a nonempty convex set. The subspace parallel to the set C , denoted by $\text{par } C$, is defined by

$$\text{par } C = \text{aff } C - x \quad \text{for any } x \in C,$$

where $\text{aff } C$ is the affine span of C .

Definition 3.8. Suppose that the set $\mathcal{M} \subset \mathbf{R}^n$ contains the point \bar{x} . The function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth at \bar{x} relative to \mathcal{M} if \mathcal{M} is a C^2 -smooth manifold around \bar{x} and the following four properties hold:

1. (restricted smoothness) $f|_{\mathcal{M}}$ is C^2 -smooth around \bar{x} ;
2. (regularity) at every point close to \bar{x} in \mathcal{M} , the function f is subdifferentially regular and has a subgradient;
3. (normal sharpness) $N_{\mathcal{M}}(\bar{x}) = \text{par } \partial f(\bar{x})$;
4. (subgradient continuity) the subdifferential map ∂f is continuous at \bar{x} relative to \mathcal{M} .

Definition 3.9. A set $S \subset \mathbf{R}^n$ is C^2 -partly smooth at a point x relative to a set \mathcal{M} if δ_S is C^2 -partly smooth at x relative to \mathcal{M} .

Definition 3.10. Let f be a C^2 -partly smooth function at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} . Then we call \bar{x} a strong critical point of f relative to \mathcal{M} if

$$0 \in \text{ri } \partial f(\bar{x})$$

and there exists $\epsilon > 0$ such that

$$f(x) \geq f(\bar{x}) + \epsilon|x - \bar{x}|^2$$

for all points $x \in \mathcal{M}$ near \bar{x} .

Given certain assumptions, critical points of parametric partly smooth functions are stable.

Theorem 3.4 (strong critical points with parameters). Suppose the set $Q \subset \mathbf{R}^m \times \mathbf{R}^n$ is a C^2 -smooth manifold containing the point (\bar{y}, \bar{x}) and satisfies the condition

$$(w, 0) \in N_Q(\bar{y}, \bar{x}) \Rightarrow w = 0.$$

For each $y \in \mathbf{R}^m$ we define the set

$$Q_y = \{x \in \mathbf{R}^n : (y, x) \in Q\}.$$

Given any function $p : \mathbf{R}^m \times \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$, define a function $p_y : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ by

$$p_y(x) = p(y, x), \text{ for } y \in \mathbf{R}^m \text{ and } x \in \mathbf{R}^n.$$

Suppose the function p is C^2 -partly smooth relative to Q . If \bar{x} is a strong critical point of the function $p_{\bar{y}}$ relative to the set $Q_{\bar{y}}$, then there are open neighborhoods $U \subset \mathbf{R}^n$ of \bar{x} and $V \subset \mathbf{R}^m$ of \bar{y} and a C^1 -smooth function $\Psi : V \rightarrow U$ satisfying $\Psi(\bar{y}) = \bar{x}$, and with the following properties, for all vectors $y \in V$:

1. for all vectors $y \in V$ the set $Q_y \cap U$ is a C^2 -smooth manifold;
2. for all vectors $y \in V$ the function p_y is C^2 -partly smooth relative to $Q_y \cap U$;

3. the function $p_y|_{Q_y \cap U}$ has a unique critical point $\Psi(y)$;
4. $\Psi(y)$ is a strong critical point of the function p_y relative to $Q_y \cap U$.

Proof. See [17, Thms. 5.2, 5.3, 5.7]. □

The concept of prox-regularity extends properties of convexity to a broader class of functions. It is essential for partly smooth functions to locally identify their manifolds uniquely.

Definition 3.11. A function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is prox-regular at a point \bar{x} for a subgradient $\bar{v} \in \partial f(\bar{x})$ if f is finite at \bar{x} , locally lower semi-continuous at \bar{x} , and there exist $r > 0$ and $\epsilon > 0$ such that

$$f(x') > f(x) + \langle v, x' - x \rangle - \frac{r}{2}|x' - x|^2 \text{ for } x' \neq x \text{ when}$$

$$|x' - \bar{x}| < \epsilon, |x - \bar{x}| < \epsilon, |f(x) - f(\bar{x})| < \epsilon, |v - \bar{v}| < \epsilon, v \in \partial f(x).$$

More precisely, we say f is prox-regular at \bar{x} for \bar{v} with respect to ϵ and r . Further, f is prox-regular at \bar{x} if it is prox-regular at \bar{x} for every $\bar{v} \in \partial f(\bar{x})$. A set S is prox-regular at \bar{x} for $\bar{v} \in N_S(\bar{x})$ if its indicator function δ_S is prox-regular at \bar{x} for $\bar{v} \in \partial \delta_S(\bar{x})$.

Proposition 3.3. Suppose the set $S \subset \mathbf{R}^n$ is closed. Then S is prox-regular at the point $\bar{x} \in S$ if and only if the projection mapping P_S is single-valued near \bar{x} .

Proof. See [40, Thm. 1.3]. □

Definition 3.12. For a proper lower semicontinuous function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ and parameter value $\lambda > 0$, the proximal mapping $P_\lambda f$ is defined by

$$P_\lambda f(x) := \operatorname{argmin}_w \left\{ f(w) + \frac{1}{2\lambda}|w - x|^2 \right\}.$$

Definition 3.13. For $\epsilon > 0$, the f -attentive ϵ -localization of ∂f around (\bar{x}, \bar{v}) is a (generally set-valued) mapping $T : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ defined by

$$T(x) = \begin{cases} \{ v \in \partial f(x) \mid |v - \bar{v}| < \epsilon \}, & \text{if } |x - \bar{x}| < \epsilon \text{ and } |f(x) - f(\bar{x})| < \epsilon, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 3.14. For a function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$, a set $V \subset \mathbf{R}^n$ is called an f -attentive neighborhood of \bar{x} if there exists a $\delta > 0$ such that

$$\{ x \in \mathbf{R}^n \mid |x - \bar{x}| < \delta, |f(x) - f(\bar{x})| < \delta \} \subset V.$$

Definition 3.15. A function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is subdifferentially continuous at a point \bar{x} for \bar{v} , where $\bar{v} \in \partial f(\bar{x})$, if for every $\delta > 0$ there exists $\epsilon > 0$ such that $|f(x) - f(\bar{x})| < \delta$ whenever $|x - \bar{x}| < \epsilon$ and $|v - \bar{v}| < \epsilon$ with $v \in \partial f(x)$.

Proposition 3.4 (prox-regularity and proximal mapping). Suppose that the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is prox-regular at $\bar{x} = 0$ for $\bar{v} = 0$ with respect to ϵ and r . In particular suppose f satisfies the following assumption:

$$\begin{cases} f \text{ is locally lower semicontinuous at } 0 \text{ with } f(0) = 0, \text{ and} \\ r > 0 \text{ is such that } f(x) > -\frac{r}{2}|x|^2 \text{ for all } x \neq 0 \end{cases}$$

(which implies that $P_\lambda f(0) = \{0\}$ when $\lambda \in (0, \frac{1}{r})$). Let T be the f -attentive ϵ -localization T of ∂f around $(0, 0)$. Then for each $\lambda \in (0, \frac{1}{r})$ there is a neighborhood X of $\bar{x} = 0$ such that, on X , the mapping $P_\lambda f$ is single-valued and continuous and

$$P_\lambda f(x) = (I + \lambda T)^{-1}(x).$$

Proof. See [37, Thm. 4.4]. □

Lemma 3.1. Suppose that the function f is extended- C^2 -smooth at \bar{x} . Then f is subdifferentially regular, prox-regular, and subdifferentially continuous at \bar{x} .

Proof. See [37, Ex. 2.8]. □

Note: Function f being prox-regular at \bar{x} doesn't imply that f is subdifferentially regular at \bar{x} . Here is an example. Let $f(x, y) = (x - |y|)^{\frac{1}{3}}$. Since there is no subgradient at $(0, 0)$, then f is prox-regular there. However, $\text{epi } f$ is not Clarke regular at $(0, 0, 0)$ which implies that f is not subdifferentially regular at $(0, 0)$.

3.4 Identification for functions

A partly smooth function has a smooth structure on its corresponding manifold. [12, Thm. 5.3] gives a nice “identification” property for partly smooth, prox-regular functions. Though this theorem is true, its proof is flawed because it depends on the assumption that the prox-regularity of a function implies the prox-regularity of its epigraph. We will prove this theorem by using proximal mappings in this section. First, let's see an example which shows that the prox-regularity of a function isn't equivalent to the prox-regularity of its epigraph.

Example 3.1 (prox-regularity of functions versus epigraphs). *Consider the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(2^n) = \sqrt{2^n}$ for any $n \in \mathbf{Z}$, f affine on $[2^n, 2^{n+1}]$, $f(0) = 0$ and $f(x) = f(-x)$ for any x . First note that*

$$\begin{aligned}\partial f(\pm 2^n) &= \left\{ \pm \frac{1}{\sqrt{2^{n-1}} + \sqrt{2^n}}, \pm \frac{1}{\sqrt{2^n} + \sqrt{2^{n+1}}} \right\}, \\ \partial f(\pm x) &= \pm \frac{1}{\sqrt{2^n} + \sqrt{2^{n+1}}}, \quad x \in (2^n, 2^{n+1}), \\ \partial f(0) &= (-\infty, +\infty).\end{aligned}$$

Next, we are going to prove that f is prox-regular at 0 for any $v \in \partial f(0)$. It is equivalent to show that there exist $\epsilon > 0$ and $r > 0$ such that

$$f(x') > f(x) + \langle u, x' - x \rangle - \frac{r}{2} |x' - x|^2 \quad \text{for } x \neq x' \text{ when}$$

$$|x'| < \epsilon, |x| < \epsilon, |f(x)| < \epsilon, |v - u| < \epsilon, u \in \partial f(x).$$

For any $x \rightarrow 0$ and $u \in \partial f(x)$ we have that $|u| \rightarrow +\infty$. Since $|v - u| < \epsilon$ and $|x| < \epsilon$, then x has to be 0 when ϵ is small. Hence we just have to prove

$$f(x') > \langle u, x' \rangle - \frac{r}{2}|x'|^2.$$

By the definition of f , we know that $f(x') > \langle \frac{1}{\sqrt{2^{n-1} + \sqrt{2^n}}}, x' \rangle$. Thus f is prox-regular at 0. However, $\text{epi } f$ is not prox-regular at $(0, 0)$. If so, there should be a neighborhood V of $(0, 0)$ such that the projection mapping $P_{\text{epi } f}$ is single-valued on V , by Proposition 3.3. However, $P_{\text{epi } f}$ is not single-valued around $(\pm 2^n, \sqrt{2^n})$ for any $n \in \mathbf{Z}$. Thus $\text{epi } f$ is not prox-regular at $(0, 0)$.

Lemma 3.2. Suppose the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} . Then $f + \delta_{\mathcal{M}}$ is prox-regular at \bar{x} and $\partial f(\bar{x}) \subset \partial(f + \delta_{\mathcal{M}})(\bar{x})$.

Proof. Let h be any C^2 -smooth representative of f around \bar{x} . Since $f \leq f + \delta_{\mathcal{M}} = h + \delta_{\mathcal{M}}$ and $f + \delta_{\mathcal{M}}$ is extended- C^2 -smooth at \bar{x} , so $f + \delta_{\mathcal{M}}$ is prox-regular at \bar{x} by Lemma 3.1, and $\hat{\partial} f(\bar{x}) \subset \hat{\partial}(f + \delta_{\mathcal{M}})(\bar{x})$. The result follows since f and $h + \delta_{\mathcal{M}}$ are both regular at \bar{x} . \square

Proposition 3.5 (subdifferential smoothness). Suppose that the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} with $\bar{y} \in \text{ri } \partial f(\bar{x})$. Let h be any C^2 -smooth representative of f around \bar{x} . Then

$$\text{gph } \partial f \cap (\mathcal{M} \times \mathbf{R}^n) = \text{gph } (\nabla h + N_{\mathcal{M}}) \cap (\mathcal{M} \times \mathbf{R}^n) \text{ locally around } (\bar{x}, \bar{y}).$$

Proof. According to [17, Prop. 2.4], we know

$$\partial f(x) \subset \text{aff } \partial f(x) = \nabla h(x) + N_{\mathcal{M}}(x)$$

for any x close to \bar{x} in \mathcal{M} . Thus

$$\text{gph } \partial f \cap (\mathcal{M} \times \mathbf{R}^n) \subset \text{gph } (\nabla h + N_{\mathcal{M}}) \cap (\mathcal{M} \times \mathbf{R}^n) \text{ locally around } (\bar{x}, \bar{y}).$$

Next, we claim the reverse inclusion: given that (x, y) is close to (\bar{x}, \bar{y}) , then $y \in \nabla h(x) + N_{\mathcal{M}}(x)$ implies $y \in \partial f(x)$. If this is not true, then there exist sequences $x_n \rightarrow \bar{x}$ in \mathcal{M} and $y_n \in \text{aff } \partial f(x_n) \rightarrow \bar{y}$ with $y_n \notin \partial f(x_n)$. Since f is regular at x when x is close to \bar{x} in \mathcal{M} , then $\partial f(x)$ is closed and convex. According to the separation theorem, for all large n there exists a unit vector $z_n \in \text{par } \partial f(x_n) = N_{\mathcal{M}}(x_n)$ such that

$$\langle z_n, u \rangle \geq \langle z_n, y_n \rangle$$

for all $u \in \partial f(x_n)$. Passing to a subsequence if necessary, we can assume that z_n approaches a unit vector z . Since ∂f is continuous at \bar{x} relative to \mathcal{M} , then $\partial f(x_n)$ converges to $\partial f(\bar{x})$. Also, $N_{\mathcal{M}}(x_n)$ converges to $N_{\mathcal{M}}(\bar{x})$. As a result, we have

$$z \in N_{\mathcal{M}}(\bar{x}) \text{ and } \langle z, u \rangle \geq \langle z, \bar{y} \rangle$$

for any $u \in \partial f(\bar{x})$. To see this, choose $u_n \rightarrow u$ satisfying $u_n \in \partial f(x_n)$, note $\langle z_n, u_n \rangle \geq \langle z_n, y_n \rangle$, and take limits. This shows that \bar{y} is separated from the convex set $\partial f(\bar{x})$ in its affine span. However, that contradicts the fact that $\bar{y} \in \text{ri } \partial f(\bar{x})$. The result follows. \square

Corollary 3.2 (set version of subdifferential smoothness). *Suppose that a set $S \subset \mathbf{R}^n$ is partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} with $\bar{y} \in \text{ri } N_S(\bar{x})$. Then*

$$\text{gph } N_S \cap (\mathcal{M} \times \mathbf{R}^n) = \text{gph } N_{\mathcal{M}} \cap (\mathcal{M} \times \mathbf{R}^n) \text{ locally around } (\bar{x}, \bar{y}).$$

Proposition 3.6 (extended-smooth reduction). *Suppose that the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} with $0 \in \text{ri } \partial f(\bar{x})$, and f is prox-regular at \bar{x} for 0. Then if $\lambda > 0$ is sufficiently small, there exists a neighborhood V of \bar{x} on which the proximal mappings $P_{\lambda} f$ and $P_{\lambda}(f + \delta_{\mathcal{M}})$ agree.*

Proof. By changing variables, we can assume $\bar{x} = 0$ without loss of generality. According to Lemma 3.2, we know that $f + \delta_M$ is also prox-regular at 0. We can choose ϵ and r such that f and $f + \delta_M$ are both prox-regular at 0 for 0 with respect to r and ϵ , particularly with the assumption in Proposition 3.4 holding. Let T be the f -attentive ϵ -localization of ∂f around $(\bar{x}, 0)$. For any $\lambda \in (0, 1/r)$ there exists a neighborhood X of $\bar{x} = 0$ such that both $P_\lambda f$ and $P_\lambda(f + \delta_M)$ are single-valued and continuous, by Proposition 3.4. In order to prove this proposition, it is sufficient to prove that for any $x_n \rightarrow \bar{x}$ we have $P_\lambda f(x_n) = P_\lambda(f + \delta_M)(x_n)$ for large n . Let h be any C^2 -smooth representative of f on M , and define

$$w_n = P_\lambda(f + \delta_M)(x_n) = \operatorname{argmin}_x \left\{ h(x) + \delta_M(x) + \frac{1}{2\lambda} |x - x_n|^2 \right\} \in M.$$

Since the assumption in Proposition 3.4 holds for $f + \delta_M$, we have $P_\lambda(f + \delta_M)(\bar{x}) = \bar{x}$. Moreover, the continuity of $P_\lambda(f + \delta_M)$ implies $w_n \rightarrow \bar{x}$. Consequently, $x_n - w_n \rightarrow 0$. Since w_n minimizes $h(x) + \delta_M(x) + \frac{1}{2\lambda} |x - x_n|^2$, then

$$0 \in \partial(h(w_n) + \delta_M(w_n) + \frac{1}{2\lambda} |w_n - x_n|^2) = \nabla h(w_n) + N_M(w_n) + \frac{1}{\lambda} (w_n - x_n)$$

or equivalently

$$\frac{1}{\lambda} (x_n - w_n) \in \nabla h(w_n) + N_M(w_n).$$

Since $0 \in \operatorname{ri} \partial f(\bar{x})$ and $\frac{1}{\lambda} (x_n - w_n) \rightarrow 0$, then by Proposition 3.5 we know

$$\frac{1}{\lambda} (x_n - w_n) \in \partial f(w_n) \text{ for large } n,$$

which also implies

$$\frac{1}{\lambda} (x_n - w_n) \in T(w_n) \text{ for large } n,$$

since $w_n \rightarrow \bar{x}$ in M , so $f(w_n) \rightarrow f(\bar{x})$. Thus

$$x_n \in (I + \lambda T)(w_n) \text{ for all large } n,$$

from which we get

$$w_n \in (I + \lambda T)^{-1}(x_n) = P_\lambda f(x_n) \text{ for all large } n$$

by Proposition 3.4. Hence $P_\lambda f(x_n) = P_\lambda(f + \delta_M)(x_n)$ for all large n . \square

If $0 \in \text{ri } \partial f(\bar{x})$ doesn't hold, the above result can fail. Here is an example.

Example 3.2. Define the function f as follows:

$$f(x) = \begin{cases} +\infty, & x \in (-\infty, 0), \\ 0, & x \in [0, \infty). \end{cases}$$

It is easy to see that f is prox-regular at x for all $x \in [0, \infty)$, and partly smooth at 0 relative to $\mathcal{M} = \{0\}$. Since $\partial f(0) = (-\infty, 0]$, then 0 doesn't lie in the interior of $\partial f(0)$.

For any small $\lambda > 0$,

$$P_\lambda f(x) = \operatorname{argmin}_w \{ f(w) + \frac{1}{2\lambda} |x - w|^2 \} = x \text{ for all } x > 0.$$

Corollary 3.3 (set version of extended-smooth reduction). *Let \mathcal{M} be a C^2 -smooth manifold around a point \bar{x} . Suppose that a set S is partly smooth at \bar{x} relative to \mathcal{M} , and that S is prox-regular at \bar{x} for $\bar{v} \in \text{ri } N_S(\bar{x})$. Suppose that $\lambda > 0$ is sufficiently small. Then for x sufficiently close to \bar{x} , the projections $P_S(x + \lambda \bar{v})$ lies in \mathcal{M} .*

Proof. Apply Proposition 3.6 to $f(x) = \delta_S(x) - \langle \bar{v}, x \rangle$. \square

Corollary 3.4 (active manifold as proximal range). *Under the same assumption as Proposition 3.6, the set $P_\lambda f(V)$ is a neighborhood of \bar{x} in \mathcal{M} for any sufficiently small neighborhood V of \bar{x} .*

Proof. By Proposition 3.6, it is sufficient to prove that for any $x_n \rightarrow \bar{x}$ in \mathcal{M} , there exists $w_n \rightarrow \bar{x}$ with $P_\lambda f(w_n) = x_n$ for large n . Since f is partly smooth at \bar{x} relative

to \mathcal{M} , then there exists $y_n \in \partial f(x_n) \rightarrow 0$. For large n , we have $y_n \in T(x_n)$. So $x_n + \lambda y_n \in (I + \lambda T)(x_n)$, which implies $x_n = P_\lambda f(x_n + \lambda y_n)$. Let $w_n = x_n + \lambda y_n$. The result follows. \square

Corollary 3.5 (set version of active manifold as proximal range). *Under the same assumption as in Corollary 3.3, for any sufficiently small neighborhood V of \bar{x} , the projection $P_S(V + \lambda \bar{v})$ is a neighborhood of \bar{x} in \mathcal{M} .*

Proof. Apply Corollary 3.4 to $f(x) = \delta_S(x) - \langle \bar{v}, x \rangle$. \square

Theorem 3.5 (identification). *Let the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ be C^2 -partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} and prox-regular at \bar{x} for $\bar{y} \in \text{ri } \partial f(\bar{x})$. Suppose $x_k \rightarrow \bar{x}$ and $f(x_k) \rightarrow f(\bar{x})$. Then*

$$x_k \in \mathcal{M} \text{ for all large } k$$

if and only if

$$\text{dist}(\bar{y}, \partial f(x_k)) \rightarrow 0.$$

Proof. By subtracting an affine function from f and changing variables, we can assume $\bar{x} = 0$, $\bar{y} = 0$ and $f(\bar{x}) = 0$ without loss of generality. Since f is prox-regular at 0 for 0, then there exist $\epsilon > 0$ and $r > 0$ such that

$$f(x') > f(x) + \langle v, x' - x \rangle - \frac{r}{2}|x - x'|^2 \text{ for } x' \neq x$$

whenever $|x| < \epsilon$, $|f(x)| < \epsilon$, $|x'| < \epsilon$, $|v| < \epsilon$, and $v \in \partial f(x)$. Letting $x = 0$, $v = 0$, we have

$$f(x') > -\frac{r}{2}|x'|^2$$

for any $|x'| < \epsilon$ and $x' \neq x$. Since we are interested only in the local geometry of $\text{epi } f$ around $(0, 0)$, then we can add to f the indicator of a compact neighborhood $\bar{B}_{\frac{\epsilon}{2}}(0)$, which is a closed ball centered at 0 with radius $\frac{\epsilon}{2}$, to make the

assumption in Proposition 3.4 hold for f : if this proposition is true for $f + \delta_{\bar{B}_{\frac{\epsilon}{2}}(0)}$, it is also true for f . To sum up, we can assume f satisfies the assumption in Proposition 3.4 without loss of generality. Fix λ such that Proposition 3.6 holds for f . Let T be the f -attentive ϵ -localization of ∂f . If $\text{dist}(0, \partial f(x_k)) \rightarrow 0$, then there exists a sequence of $y_k \rightarrow 0$ with $y_k \in \partial f(x_k)$. Then we have

$$\frac{1}{\lambda}((x_k + \lambda y_k) - x_k) \in \partial f(x_k),$$

which implies

$$\frac{1}{\lambda}((x_k + \lambda y_k) - x_k) \in T(x_k) \text{ for large } k.$$

Thus

$$x_k + \lambda y_k \in (I + \lambda T)(x_k) \text{ for large } k.$$

Consequently

$$x_k = (I + \lambda T)^{-1}(x_k + \lambda y_k) = P_{\lambda} f(x_k + \lambda y_k) \in \mathcal{M} \text{ for large } k,$$

by Proposition 3.6. Thus the result follows since the converse is immediate by partial smoothness. \square

Corollary 3.6 (identification for sets). *Let the set S be C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} and prox-regular there for $\bar{n} \in \text{ri } N_S(\bar{x})$. If the sequence $\{x_k\} \in S$ satisfies $x_k \rightarrow \bar{x}$, then*

$$\text{dist}(\bar{n}, N_S(x_k)) \rightarrow 0 \Leftrightarrow x_k \in \mathcal{M} \text{ for all large } k.$$

Proof. The result follows by applying Theorem 3.5 to the indicator function δ_S . \square

Corollary 3.7 (uniqueness of active manifold). *Consider a set S that is prox-regular at a point \bar{x} for $\bar{n} \in \text{ri } N_S(\bar{x})$ and C^2 -partly smooth there relative to each of the two C^2 -smooth manifolds \mathcal{M}_1 and \mathcal{M}_2 . Then near \bar{x} we have $\mathcal{M}_1 \equiv \mathcal{M}_2$.*

Proof. If this is not true, then there exists a sequence of points x_k converging to \bar{x} such that $x_k \in \mathcal{M}_1 \setminus \mathcal{M}_2$. Since S is partly smooth relative to \mathcal{M}_1 , then the normal cone $N_S(x_k) \rightarrow N_S(\bar{x})$. Hence $\text{dist}(\bar{n}, N_S(x_k)) \rightarrow 0$. Applying Corollary 3.6 to δ_S with $\mathcal{M} \equiv \mathcal{M}_2$ implies $x_k \in \mathcal{M}_2$ for all large k , which is contradictory to $x_k \notin \mathcal{M}_2$. Thus the result follows. \square

The definition of strong critical points demands quadratic growth along the manifold. Under the assumption of prox-regularity, strong critical points of such functions are actually locally quadratic minimizers or “strict local minimizers of order two” in the terminology of [10]. [12, Thm. 6.2] gives a proof, requiring such functions to be prox-regular at the local minimizer. In this chapter, we use another approach to prove this with a more natural, slightly weaker assumption, requiring only that such functions be prox-regular at the minimizer for the subgradient 0.

Proposition 3.7 (Sufficient optimality conditions). *Suppose the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} and prox-regular there for $0 \in \text{ri } \partial f(\bar{x})$. Then the following hold:*

1. \bar{x} is a strict local minimizer of the restricted function $f|_{\mathcal{M}} \Leftrightarrow \bar{x}$ is in fact an unconstrained strict local minimizer of f .
2. \bar{x} is a strong critical point of f relative to $\mathcal{M} \Leftrightarrow f$ grows at least quadratically near \bar{x} .

Proof. One direction of both cases is obvious. Let’s prove the other direction. First we are going to prove that \bar{x} being a strict local minimizer of the restricted function $f|_{\mathcal{M}}$ is equivalent to \bar{x} being an unconstrained strict local minimizer of f . Without loss of generality, let $\bar{x} = 0$, $f(\bar{x}) = 0$, and f satisfy the assumption

in Proposition 3.4. We are going to prove this proposition by contradiction. Suppose there exists a sequence $x_k \notin \mathcal{M} \rightarrow \bar{x}$ with

$$f(x_k) \leq f(\bar{x}) \text{ for all } k.$$

For large k we know that x_k lies in the f -attentive neighborhood of \bar{x} in Proposition 3.6. Hence $x_k \neq y_k = P_\lambda f(x_k) \in \mathcal{M}$ and $y_k \rightarrow P_\lambda f(\bar{x}) = \bar{x}$. Then we have

$$\begin{aligned} f(\bar{x}) &\geq f(x_k) \\ &\geq \min_w \left\{ f(w) + \frac{1}{2\lambda} |x_k - w|^2 \right\} \\ &= f(y_k) + \frac{1}{2\lambda} |y_k - x_k|^2 \\ &> f(\bar{x}) + \frac{1}{2\lambda} |y_k - x_k|^2. \end{aligned}$$

Consequently, we get a contradiction:

$$0 > \frac{1}{2\lambda} |y_k - x_k|^2.$$

Next we are going to prove part 2. Since f grows quadratically at \bar{x} relative to \mathcal{M} , then there exists a $\delta > 0$ such that $f(x) > \delta |x - \bar{x}|^2$ around \bar{x} relative to \mathcal{M} . Define h by $h(x) = f(x) - \delta |x - \bar{x}|^2$. Since $\delta |x - \bar{x}|^2$ is C^2 -smooth, then h is also prox-regular at \bar{x} for $0 \in \text{ri } \partial h(\bar{x})$ and partly smooth at \bar{x} relative to \mathcal{M} . Moreover, we know that $h(x) > h(\bar{x})$ locally around \bar{x} restricted to \mathcal{M} . According to case 1, we know that $h(x) > h(\bar{x})$ locally around \bar{x} . Then the second part follows. \square

3.5 Calculation of generalized Hessian mappings

In general it may be hard to compute the generalized Hessian mapping. Our goal is to analyze the generalized Hessian mapping in the easier special case of partly smooth and prox-regular functions. Given these assumptions plus

subdifferential continuity property, Theorem 3.5 guarantees that the local geometry of $\text{gph } \partial f$ is determined by $f|_{\mathcal{M}}$. This smooth structure simplifies the computation of the generalized Hessian mapping and also gives a geometrical explanation of the second condition in Theorem 3.7.

Proposition 3.8 (subdifferential localization and active manifolds). *Suppose that the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} , and both prox-regular and subdifferentially continuous at \bar{x} for $\bar{y} \in \text{ri } \partial f(\bar{x})$. Then*

$$\text{gph } \partial f \subset \mathcal{M} \times \mathbf{R}^n$$

locally around (\bar{x}, \bar{y}) .

Proof. Since f is subdifferentially continuous at \bar{x} for $\bar{y} \in \partial f(\bar{x})$, then $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$ with $y_n \in \partial f(x_n)$ implies $f(x_n) \rightarrow f(\bar{x})$. According to Theorem 3.5, we know $x_n \in \mathcal{M}$ for all large n , so the result follows. \square

Corollary 3.8 (smooth reduction for subdifferential localization). *Suppose the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} , and both prox-regular and subdifferentially continuous at \bar{x} for $\bar{y} \in \text{ri } \partial f(\bar{x})$. Let h be any C^2 -smooth representative of f around \bar{x} . Then*

$$\text{gph } \partial f = \text{gph } (\nabla h + N_{\mathcal{M}}) \cap (\mathcal{M} \times \mathbf{R}^n) = \text{gph } \partial(f + \delta_{\mathcal{M}})$$

locally around (\bar{x}, \bar{y}) .

Proof. This result is easily derived from Propositions 3.5 and 3.8. \square

The following result gives a formula for the generalized Hessian mapping for partly smooth and prox-regular functions.

Theorem 3.6 (generalized and covariant Hessians). *Suppose that the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} and both prox-regular and subdifferentially continuous at \bar{x} for $0 \in \text{ri } \partial f(\bar{x})$. Then*

$$\partial^2 f(\bar{x}|0)(w) = \begin{cases} \nabla_{\mathcal{M}}^2 f(\bar{x})w + N_{\mathcal{M}}(\bar{x}) & \text{for } w \in T_{\mathcal{M}}(\bar{x}), \\ \emptyset & \text{for } w \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

Proof. According to Corollary 3.8, we have that $\partial^2 f(\bar{x}|0) = \partial^2(f + \delta_{\mathcal{M}})(\bar{x}|0)$. Then by Theorem 3.3, the result follows. \square

Corollary 3.9. *Suppose that the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} and both prox-regular and subdifferentially continuous at \bar{x} for $\bar{v} \in \text{ri } \partial f(\bar{x})$. Let $\tilde{f}(x) = f(x) - \langle \bar{v}, x \rangle$. Then*

$$\partial^2 f(\bar{x}|\bar{v})(w) = \begin{cases} \nabla_{\mathcal{M}}^2 \tilde{f}(\bar{x})w + N_{\mathcal{M}}(\bar{x}) & \text{for } w \in T_{\mathcal{M}}(\bar{x}), \\ \emptyset & \text{for } w \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

Proof. First note that

$$\partial^2 \tilde{f}(\bar{x}|0)(w) = \partial^2(f - \langle \bar{v}, \cdot \rangle)(\bar{x}|0)(w) = \partial^2 f(\bar{x}|\bar{v})(w) \text{ for all } w.$$

Furthermore, we know \tilde{f} is partly smooth at \bar{x} relative to \mathcal{M} and both prox-regular and subdifferentially continuous at \bar{x} for $0 \in \text{ri } \partial \tilde{f}(\bar{x})$. According to Theorem 3.6, we have

$$\partial^2 f(\bar{x}|\bar{v})(w) = \partial^2 \tilde{f}(\bar{x}|0)(w) = \begin{cases} \nabla_{\mathcal{M}}^2 \tilde{f}(\bar{x})w + N_{\mathcal{M}}(\bar{x}) & \text{for } w \in T_{\mathcal{M}}(\bar{x}), \\ \emptyset & \text{for } w \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

\square

Without subdifferential continuity, the above result will fail in general.

Example 3.3. Define the function $f : \mathbf{R} \rightarrow \mathbf{R}$ as follows:

$$f(x) = \begin{cases} 1, & x \in (-\infty, 0), \\ x, & x \in [0, \infty). \end{cases}$$

It is easy to check that f is prox-regular at 0 with $0 \in \text{ri } \partial f(0)$ and partly smooth relative to the manifold $\mathcal{M} = \{0\}$. However, the function f is not subdifferentially continuous at 0 for $0 \in \partial f(0)$. Then $\text{gph } \partial f \neq \text{gph } \partial(f + \delta_{\mathcal{M}})$ locally around $(0, 0)$.

Note: Corollary 3.1 gives a more concrete description of the generalized Hessian in terms of a smooth representative of f and smooth equations for \mathcal{M} . Next, we will use Theorem 3.6 to calculate the generalized Hessian mapping for the maximum eigenvalue function. We will use \mathcal{U} -Lagrangian in this example. Let's introduce the definition first (cf. [26]).

Definition 3.16. Suppose that a convex function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at a point \bar{x} relative to a C^2 -smooth manifold \mathcal{M} . Let $\mathcal{U}(\bar{x}) = T_{\mathcal{M}}(\bar{x})$ and $\mathcal{V}(\bar{x}) = N_{\mathcal{M}}(\bar{x})$. Given $\bar{g} \in \partial f(\bar{x})$, the \mathcal{U} -Lagrangian of f is the function $L_{\mathcal{U}}^f(\bar{x}; \bar{g}; \cdot) : \mathcal{U}(\bar{x}) \rightarrow \mathbf{R}$ defined by

$$L_{\mathcal{U}}^f(\bar{x}; \bar{g}; u) = \inf_{v \in \mathcal{V}(\bar{x})} \{ f(\bar{x} + u + v) - \bar{g}^T v \}.$$

Let $g_{\mathcal{U}}(\bar{x}) = \nabla_{\mathcal{M}} f(\bar{x})$. According to [26], we have $g_{\mathcal{U}}(\bar{x}) = \nabla_u L_{\mathcal{U}}^f(\bar{x}; g_{\mathcal{U}}(\bar{x}); 0)$ and $\nabla_{\mathcal{M}}^2 f(\bar{x}) = \nabla_{uu}^2 L_{\mathcal{U}}^f(\bar{x}; g_{\mathcal{U}}(\bar{x}); 0)$.

Example 3.4. Let \mathbf{S}^n be the space consisting of the n -by- n real symmetric matrices. Suppose the function $\lambda_1 : \mathbf{S}^n \rightarrow \mathbf{R}$ maps every real symmetric matrix to its maximum eigenvalue. According to [17, Exp. 3.6], we have the following results:

1. λ_1 is partly smooth relative to the manifold

$$\mathcal{M}_m = \{ X \in \mathbf{S}^n : \lambda_1(X) \text{ has multiplicity } m \} \quad (1 \leq m \leq n).$$

2. λ_1 is a finite-valued convex function. Hence λ_1 is prox-regular and subdifferentially continuous everywhere.
3. There is an n -by- m matrix $Q(X)$, depending continuously on $X \in \mathcal{M}_m$, whose columns are a basis for the eigenspace of X corresponding to $\lambda_1(X)$, and then we have

$$\begin{aligned} N_{\mathcal{M}_m}(X) &= Q(X)\{W \in \mathbf{S}^m : \text{trace } W = 0\}Q(X)^T, \\ \partial\lambda_1(X) &= Q(X)\{W \in \mathbf{S}_+^m : \text{trace } W = 1\}Q(X)^T, \end{aligned}$$

where \mathbf{S}_+^m denotes the set of the positive semidefinite matrices.

Now suppose $\bar{X} \in \mathcal{M}_m$ and $\bar{G} \in \text{ri } \partial\lambda_1(\bar{X})$. Let $\mu(X) = \lambda_1(X) - \langle \bar{G}, X \rangle$. According to Theorem 3.6, we have

$$\partial^2 \lambda_1(\bar{X}|\bar{G})(W) = \partial^2 \mu(\bar{X}|0)(W) = \begin{cases} \nabla_{\mathcal{M}_m}^2 \mu(\bar{X})W + N_{\mathcal{M}_m}(\bar{X}) & \text{for } W \in T_{\mathcal{M}_m}(\bar{X}), \\ \emptyset & \text{for } W \notin T_{\mathcal{M}_m}(\bar{X}). \end{cases}$$

By definition,

$$L_{\mathcal{U}}^{\lambda_1}(\bar{X}; \bar{G}; U) = \inf_{V \in \mathcal{V}(\bar{X})} \{ \lambda_1(\bar{X} + U + V) - \langle \bar{G}, V \rangle \}$$

for $U \in T_{\mathcal{M}_m}(\bar{X})$, $\mathcal{V}(\bar{X}) = N_{\mathcal{M}_m}(\bar{X})$.

Since $0 \in \partial\mu(\bar{X})$, we have $\nabla_{\mathcal{M}_m} \mu(\bar{X}) = 0$ and

$$L_{\mathcal{U}}^{\mu}(\bar{X}; 0; U) = \inf_{V \in \mathcal{V}(\bar{X})} \{ \lambda_1(\bar{X} + U + V) - \langle \bar{G}, \bar{X} + U + V \rangle \}$$

for $U \in T_{\mathcal{M}_m}(\bar{X})$, $\mathcal{V}(\bar{X}) = N_{\mathcal{M}_m}(\bar{X})$.

Note that $L_{\mathcal{U}}^{\mu}(\bar{X}; 0; U) = L_{\mathcal{U}}^{\lambda_1}(\bar{X}; \bar{G}; U) - \langle \bar{G}, \bar{X} + U \rangle$. Then we have $\nabla_{\mathcal{M}_m}^2 \mu(\bar{X}) = \nabla_{UU}^2 L_{\mathcal{U}}^{\mu}(\bar{X}; 0; 0) = \nabla_{UU}^2 L_{\mathcal{U}}^{\lambda_1}(\bar{X}; \bar{G}; 0)$. According to [34, Thm. 4.12], we have

$$\nabla_{UU}^2 L_{\mathcal{U}}^{\lambda_1}(\bar{X}; \bar{G}; 0) = \text{proj}_{T_{\mathcal{M}_m}(\bar{X})} \circ H(\bar{X}, \bar{G}) \circ \text{proj}_{T_{\mathcal{M}_m}(\bar{X})}^*,$$

where $H(\bar{X}, \bar{G})$ is the symmetric linear operator on \mathbf{S}^n defined by

$$H(\bar{X}, \bar{G}) \cdot Y = \bar{G}Y[\lambda_1(\bar{X})I_n - \bar{X}]^\dagger + [\lambda_1(\bar{X})I_n - \bar{X}]^\dagger Y\bar{G} \text{ for all } Y \in \mathbf{S}^n.$$

($[\lambda_1(\bar{X})I_n - \bar{X}]^\dagger$ is the corresponding generalized inverse.)

For all $W \in T_{\mathcal{M}_m}(\bar{X})$, we have

$$H(\bar{X}, \bar{G}) \cdot W = \bar{G}W[\lambda_1(\bar{X})I_n - \bar{X}]^\dagger + [\lambda_1(\bar{X})I_n - \bar{X}]^\dagger W\bar{G}.$$

Hence we have

$$\begin{aligned} & \partial^2 \lambda_1(\bar{X}|\bar{G})(W) \\ &= \begin{cases} \bar{G}W[\lambda_1(\bar{X})I_n - \bar{X}]^\dagger + [\lambda_1(\bar{X})I_n - \bar{X}]^\dagger W\bar{G} + N_{\mathcal{M}_m}(\bar{X}) & \text{for } W \in T_{\mathcal{M}_m}(\bar{X}) \\ \emptyset & \text{for } W \notin T_{\mathcal{M}_m}(\bar{X}), \end{cases} \end{aligned}$$

by using the fact for any subspace T , the adjoint P_T^* is simply the embedding and $P_T(y) \in y + T^\perp$ for any y .

This formula, in conjunction with a suitable chain rule in [30], allows us to study the generalized Hessian of composite functions of the form $x \mapsto f(x) = \lambda_1(F(x))$ and hence second-order optimality conditions for f . Such an approach may give alternative insights into the standard second-order optimality conditions for semidefinite programs-see [45, 3]. We do not pursue that connection here.

3.6 Stability and partial smoothness

The following theorem in [38] gives a generalized Hessian characterization for tilt stability.

Theorem 3.7. *For a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ having $0 \in \partial f(\bar{x})$ and such that f is both prox-regular and subdifferentially continuous at \bar{x} for 0, the point \bar{x} gives a tilt stable local minimum of f if and only if the mapping $\partial^2 f(\bar{x}|0)$ is positive definite in the sense that*

$$\langle z, w \rangle > 0 \text{ whenever } z \in \partial^2 f(\bar{x}|0)(w), w \neq 0.$$

In this case, the mapping M from Definition 3.1 and $(\partial f)^{-1}$ have locally identical graphs around the point $(0, \bar{x})$.

Proof. See [38, Thm. 1.3]. □

With the assumption of Theorem 3.7, suppose in addition that f is C^2 -partly smooth at \bar{x} relative to the C^2 -smooth manifold \mathcal{M} . Then, by combining the result above with our Hessian calculations in the previous section, we easily deduce the equivalence of the following properties.

1. The point \bar{x} is a tilt stable local minimum of the function f .
2. The point \bar{x} is a tilt stable local minimum of the function $f + \delta_{\mathcal{M}}$.
3. The point \bar{x} is a strong critical point of f relative to \mathcal{M} .

To see this note that $\partial^2 f(\bar{x}|0) = \partial^2(f + \delta_{\mathcal{M}})(\bar{x}|0)$ by Corollary 3.8, so (a) and (b) are equivalent by Theorem 3.7. We also know that 2 is equivalent to $\partial^2(f + \delta_{\mathcal{M}})(\bar{x}|0)$ being positive definite, which is also equivalent to

$$\langle \nabla^2 L(\bar{x})w, w \rangle > 0 \text{ for any } 0 \neq w \in T_{\mathcal{M}}(\bar{x})$$

with L the Lagrangian of Corollary 3.1. This in turn is equivalent to \bar{x} being a strong critical point of f relative to \mathcal{M} , according to [17, p. 25]. Therefore the result follows.

With a little extra care, we can dispense with the assumption of subdifferential continuity. We use the following easy tool.

Proposition 3.9 (local minimizers and perturbation). *Suppose that the point \bar{x} gives a tilt stable local minimum of the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$. If a sequence of points $v_k \in \mathbf{R}^n \rightarrow 0$, the mapping M in Definition 3.1 satisfies*

$$M(v_k) \rightarrow \bar{x} \text{ and } f(M(v_k)) \rightarrow f(\bar{x}).$$

Proof. Since M is Lipschitz at 0, then $x_k := M(v_k) \rightarrow \bar{x}$. Note $f(x) > f(\bar{x})$ for any $\bar{x} \neq x \in B_\delta(\bar{x})$. Suppose that $f(x_k) \rightarrow f(\bar{x})$ is not true. Without loss of generality, we can assume that there exists an $\epsilon > 0$ such that $|f(x_k) - f(\bar{x})| > \epsilon$ for all large k . Since \bar{x} is a strict local minimizer,

$$f(x_k) > f(\bar{x}) + \epsilon.$$

Take limits on both sides. We get

$$\liminf_{k \rightarrow \infty} f(x_k) \geq f(\bar{x}) + \epsilon,$$

which is contradictory to the fact that $f(x)$ is locally lower semicontinuous at \bar{x} . Therefore $f(x_k) \rightarrow f(\bar{x})$. \square

We now have our main result.

Theorem 3.8 (strong criticality point and tilt stability). *Suppose that the function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is C^2 -partly smooth at the point \bar{x} relative to the C^2 -smooth manifold \mathcal{M} , and prox-regular at \bar{x} for $0 \in \text{ri } \partial f(\bar{x})$. Then the following are equivalent*

1. *the point \bar{x} is a tilt stable local minimum of the function f ;*
2. *the point \bar{x} is a tilt stable local minimum of the function $f + \delta_{\mathcal{M}}$;*

3. the point \bar{x} is a strong critical point of f relative to \mathcal{M} ;

4. the function f grows quadratically near \bar{x} .

Proof. By Proposition 3.7, we know $3 \Leftrightarrow 4$. Since $f + \delta_{\mathcal{M}}$ is both prox-regular and subdifferentially continuous at \bar{x} for 0 by Lemma 3.1, then we know that

$$\langle \nabla^2 L(\bar{x})w, w \rangle > 0 \text{ for any } w \in T_{\mathcal{M}}(\bar{x})$$

by Theorem 3.7. This is also equivalent to \bar{x} being a strong critical point of f relative to \mathcal{M} by previous argument. Therefore $2 \Leftrightarrow 3$. Since f is partly smooth at \bar{x} relative to \mathcal{M} and prox-regular at \bar{x} for $0 \in \text{ri } \partial f(\bar{x})$, then for any $(x_k, v_k) \rightarrow (\bar{x}, 0)$ with $v_k \in \partial f(x_k)$, we have $x_k = M(v_k)$ and $f(x_k) \rightarrow f(\bar{x})$ for large k by Proposition 3.9. Hence $x_k = M(v_k) \in \mathcal{M}$ for all large k , according to Theorem 3.5. Therefore for all large k , we have

$$\begin{aligned} M(v_k) &= \operatorname{argmin}_{|\bar{x}-x_k| \leq \delta} \{ f(x) - f(\bar{x}) - \langle v_k, x - \bar{x} \rangle \} \\ &= \operatorname{argmin}_{|\bar{x}-x_k| \leq \delta} \{ f(x) + \delta_{\mathcal{M}}(x) - f(\bar{x}) - \delta_{\mathcal{M}}(\bar{x}) - \langle v_k, x - \bar{x} \rangle \}. \end{aligned}$$

Hence the point \bar{x} gives a tilt stable local minimum of f if and only if \bar{x} gives a tilt stable local minimum of $f + \delta_{\mathcal{M}}$. In other words, we have $1 \Leftrightarrow 2$. Then the theorem follows. \square

Note: It is possible to give a direct proof of the above theorem without using generalized Hessian mappings.

One particular consequence of our main result is that, in a common concrete setting, tilt stability and quadratic growth are precisely equivalent. The relationship between these two properties has been studied in more generality in [3] and [10]. For example, [3, Thm. 5.36] shows, for a very general class of infinite-dimensional optimization problems, that tilt stability is equivalent to a certain

“uniform second-order growth” condition. By a different approach (building on [38]), [10, Cor. 39] shows, in finite dimensions, that tilt stability is equivalent to a certain second-order growth property relative to a locally dense subset of the subdifferential graph. In comparison, the equivalence resulting from our Theorem 6.3, while depending heavily on the structure of partial smoothness, is simpler.

3.7 Strong metric regularity and tilt stability

In this section, we first note that tilt stability is equivalent to “strong metric regularity” of the subdifferential.¹

Definition 3.17. *A set-valued mapping $S : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is strongly metrically regular at \bar{x} for \bar{v} if S^{-1} has a Lipschitz continuous single-valued localization around \bar{v} for \bar{x} (cf. [7]).*

Proposition 3.10. *Suppose that the function $f : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is locally lower semicontinuous at \bar{x} with $0 \in \partial f(\bar{x})$. Moreover, assume the function f is prox-regular and subdifferentially continuous at \bar{x} for $0 \in \partial f(x)$. Then the following are equivalent:*

1. *The point \bar{x} gives a tilt stable local minimum for the function f .*
2. *The point \bar{x} is a local minimizer and the subgradient mapping ∂f is strongly metrically regular at \bar{x} for 0.*

Proof. $1 \Rightarrow 2$ Suppose the point \bar{x} gives a tilt stable local minimum to the function f . Then we know

$$M(v) : v \mapsto \operatorname{argmin}_{|x-\bar{x}| \leq \delta} \{ f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle \}$$

¹After completing an initial version of this work, the authors became aware of recent work analogous to Proposition 3.10 below—see [30, Cor. 5.3].

is single-valued and Lipschitz continuous around 0 with $M(0) = \bar{x}$. Note that $M(v) = (\partial f)^{-1}(v) \cap \overline{B}_\delta(\bar{x})$ for any v close to 0. Hence ∂f is strongly metrically regular at \bar{x} for 0.

2 \Rightarrow 1 First notice that \bar{x} is a strict local minimizer. If \bar{x} is not a strict local minimizer, there exists a sequence of x_k such that $f(x_k) = f(\bar{x})$. Then $0 \in \partial f(x_k)$, which is contradictory to the the strong metrical regularity of $\partial f(x)$ at $(\bar{x}, 0)$. Since \bar{x} is a strict local minimizer, there exists a $\delta > 0$ such that $f(x) > f(\bar{x})$ for any $\bar{x} \neq x \in \overline{B}_\delta(\bar{x})$. We claim that if $v_k \rightarrow 0$ and x_k minimizes $f(x) - \langle v_k, x \rangle$ over $\overline{B}_\delta(\bar{x})$, then $x_k \rightarrow \bar{x}$. Suppose the claim is not true. Then, there exists an $\epsilon > 0$ such that there are sequences $v_k \rightarrow 0$ and x_k minimizing $f(x) - \langle v_k, x \rangle$ over $\overline{B}_\delta(\bar{x})$ with $|x_k - \bar{x}| > \epsilon$. So

$$f(x_k) - \langle v_k, x_k \rangle \leq f(\bar{x}) - \langle v_k, \bar{x} \rangle.$$

Without loss of generality, choose a subsequence of x_r which converges to \hat{x} . Since f is locally lower semicontinuous at \bar{x} , we have

$$f(\hat{x}) \leq f(\bar{x})$$

by taking limits on both sides. We get a contradiction. Next we define the following mapping

$$M(v) : v \mapsto \operatorname{argmin}_{|x-\bar{x}| \leq \delta} \{ f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle \} \text{ with } M(0) = \bar{x}.$$

According to the claim, we know that $M(v)$ should lie in the interior of $B_\delta(\bar{x})$ for small v . Therefore $M(v)$ is also a critical point of $f(x) - \langle v, x \rangle$ for all small v . Since ∂f is strongly metrically regular at \bar{x} for 0, then $M(v)$ is single-valued and Lipschitz continuous around 0. Therefore \bar{x} gives a tilt stable local minimum of the function f . \square

Artacho and Geoffroy [1] showed that for a proper lower semicontinuous

convex function in a Hilbert space, the strong metric regularity property of its subdifferential is equivalent to a quadratic growth condition involving the function.

Theorem 3.9. *Suppose that $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ is a proper lower semicontinuous convex function. Then ∂f is strongly metrically regular at \bar{x} for \bar{v} if and only if there exist neighborhoods X of \bar{x} and V of \bar{v} and a positive constant c such that for any $v \in V$ there is $\tilde{x} \in \mathbf{R}^n$ such that $\partial f^{-1}(v) = \{\tilde{x}\}$ and*

$$f(x) \geq f(\tilde{x}) - \langle v, \tilde{x} - x \rangle + c|x - \tilde{x}|^2 \text{ whenever } x \in X.$$

Proof. See [1, Cor. 3.9]. □

Theorem 3.8 shows that tilt stability is equivalent to a quadratic growth condition for prox-regular and partly smooth functions, which is also equivalent to the strong metric regularity of the subdifferential by Proposition 3.10. On the other hand, Theorem 3.9 implies that strong metric regularity of the subdifferential is equivalent to a quadratic growth condition for convex functions. In this sense, Proposition 3.10 is an analogue of Theorem 3.9 for a broader class of functions.

Note: After we completed this work, we became aware of concurrent work by Mordukhovich and Rockafellar [30]. As an application of the powerful second-order subdifferential calculus developed there, that paper includes in Theorem 6.1 an extension of the characterization of tilt stability in Theorem 6.1 to favorable classes of constrained optimization problems, specifically “extended nonlinear programs”. The philosophy of the current work is somewhat analogous, but concentrating instead on partly smooth functions.

Two recent papers [9] and [8] also discuss the relationship among second-order growth, tilt stability, and metric regularity of the subdifferential. [9] proves that these three notions are essentially equivalent for the general class of prox-regular and subdifferentially continuous function on finite-dimensional spaces, while [8] further studies these three topics in the infinite-dimensional setting.

CHAPTER 4

NONSMOOTHNESS AND THE BFGS METHOD

4.1 Introduction

We organize this chapter as follows. In Section 2 we demonstrate how the exact-line-search BFGS method succeeds on a representative convex nonsmooth function. We also provide numerical evidence for linear convergence of an inexact-line-search BFGS on the same example. In Section 3 we present an illustrative proof that the inexact-line-search BFGS method cannot stall at a spurious limit point when applied to a representative nonsmooth function without any stationary points. In Section 4 we give an example of how the inexact-line-search BFGS method can converge to a limit point with descent directions. (This example does not disprove the challenge question from [19], since the limit point is nonetheless Clarke stationary.) In Section 5 we discuss possible reasons why the line-search BFGS method seems so much more successful than the trust-region method when applied to nonsmooth functions.

In this chapter, we study the BFGS and line search algorithms described in [18] and [19]. The line-search BFGS method applied to minimize a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ iterates as follows. We use x_k , H_k , and p_k to denote the current point, the approximate inverse Hessian matrix, and the line search direction at the k th iteration. We begin with an initial point x_0 and an initial positive semidefinite matrix H_0 .

Line-search BFGS method

repeat

Stop: if f is not differential at x_k or the method generates a solution;

Search direction: $p_k = -H_k \nabla f(x_k)$;

Step length: $x_{k+1} = x_k + \alpha_k p_k$, where α_k satisfies the *Wolfe conditions*, for fixed $c_1 < c_2$ in $(0, 1)$:

$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T p_k \quad (4.1)$$

$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq c_2 \nabla f(x_k)^T p_k; \quad (4.2)$$

Gradient increment: $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$;

Inverse Hessian factor: $V_k = I - (p_k^T y_k)^{-1} p_k y_k^T$;

Inverse Hessian update: $H_{k+1} = V_k H_k V_k^T + \alpha_k (p_k^T y_k)^{-1} p_k p_k^T$;

Iteration count: $k = k + 1$;

end(repeat)

Notice that the line search employs a one-sided “weak” Wolfe condition, appropriate in the nonsmooth case, and not the more standard “strong” Wolfe condition: for more discussion, see [19, 18]. Well-known elementary properties of the BFGS method include the *secant condition*

$$s_k := x_{k+1} - x_k = H_{k+1} y_k,$$

and the fact that the matrix H_k remains positive definite. For simplicity, we use the abbreviated notation $\nabla f_k := \nabla f(x_k)$.

4.2 BFGS with exact line search

In this section, we will give a full analysis of the BFGS method with an exact line search, applied to one particular representative nonsmooth example. The exact line search step length is chosen by $\alpha_k \in \operatorname{argmin}_{\alpha} \{f(x_k + \alpha p_k)\}$. Since we set $c_1 = 0$ in this chapter, the exact line search step length satisfies the Wolfe conditions. Unlike the algorithm described in [18], the BFGS method we consider here stops whenever it encounters a nonsmooth point.

We begin with a structural property of the exact-line-search BFGS method. For simplicity, we state the result for infinite sequences of iterates.

Proposition 4.1. *Suppose the BFGS method with exact line search generates a sequence of points x_0, x_1, x_2, \dots at which the function f is smooth and noncritical. Then the following properties hold for all $k = 1, 2, 3, \dots$:*

1. $\nabla f_k^T s_{k-1} = 0$
2. $y_{k-1}^T s_k = 0$
3. $\nabla f_k^T s_k < 0$.

Conversely, suppose that f is a convex function on \mathbf{R}^2 , that $\nabla f_0^T s_0 < 0$, and that properties 1 and 2 hold for all $k = 1, 2, 3, \dots$. Then x_0, x_1, x_2, \dots is an exact-line-search BFGS sequence.

Proof. Property 1 follows immediately from the definition of the exact line search. To see property 2, note

$$y_{k-1}^T s_k = \alpha_k y_{k-1}^T p_k = -\alpha_k y_{k-1}^T H_k \nabla f_k = -\alpha_k s_{k-1}^T \nabla f_k = 0,$$

using the secant condition and property 1. Property 3 follows from the fact that H_k is positive definite.

We prove the converse by induction. Since $\nabla f(x_0)^T s_0 < 0$, we can find a positive definite matrix H_0 such that

$$p_0 = s_0 = -H_0 \nabla f_0.$$

The exact line search then seeks α_0 minimizing $f(x_0 + \alpha p_0)$. Since f is convex and $\nabla f_1^T p_0 = 0$, it follows that $\alpha_0 = 1$ is a minimizer, and hence x_1 is an exact-line-search BFGS iterate.

Now consider an exact-line-search BFGS sequence $x_0, x_1, x_2, \dots, x_k$. By the secant condition and property 1, we know

$$p_k^T y_{k-1} = -\nabla f_k^T H_k y_{k-1} = -\nabla f_k^T s_{k-1} = 0.$$

Since we are working on the space \mathbf{R}^2 and property 2 holds, the lines $\mathbf{R}p_k$ and $\mathbf{R}s_k$ must coincide, both being orthogonal to the nonzero vector y_{k-1} . Now the conditions that f is convex and $\nabla f_{k+1}^T s_k = 0$ imply that $\alpha = \alpha_k$ minimizes the function $f(x_k + \alpha p_k)$ in the line search. Hence x_{k+1} is a valid next iterate for the method. The proposition follows. \square

A parametrized example

We next consider a simple but illustrative nonsmooth example on \mathbf{R}^2 . This function has a global minimizer at zero and is nonsmooth at every point on one axis (and indeed is “partly smooth” [17] relative to that axis). If we initialize appropriately, the algorithm will generate a sequence of points alternating between

two parabolas and converging linearly to the optimal solution. In all other cases, the algorithm will terminate at a nonsmooth point after finitely many iterations.

Proposition 4.2. *Consider the exact-line-search BFGS method, applied to minimize the function*

$$f(u, v) = \max\{u^2 + v, u^2 - av\}$$

for some fixed parameter $a > 0$, initialized with

$$(u_0, v_0) = \left(1, \frac{2}{a^2 + 3a + 1}\right) \text{ and } H_0 = \begin{pmatrix} \frac{a}{2(a+1)} & 0 \\ 0 & \frac{2}{(a+1)^2} \end{pmatrix}.$$

The iterates converge linearly to the unique global minimizer zero, with rate $\rho = \frac{a}{(a+1)^2}$, and oscillate between the two parabolas

$$v = \frac{2}{a^2 + 3a + 1}u^2 \text{ and } v = -\frac{2a}{a^2 + 3a + 1}u^2.$$

Explicitly, the iterates are given by

$$(u_{2k}, v_{2k}) = \left(\rho^k, \frac{2\rho^{2k}}{a^2 + 3a + 1}\right), \quad (u_{2k+1}, v_{2k+1}) = \left(\frac{\rho^k}{a+1}, -\frac{2\rho^{2k+1}}{a^2 + 3a + 1}\right).$$

Moreover, the corresponding inverse Hessian approximations are, for $k > 0$,

$$\begin{aligned} H_1 &= \frac{1}{2(a^2 + a + 1)} \begin{pmatrix} 2a^2 + a & 2a(1-a)(1+a)^{-1} \\ 2a(1-a)(1+a)^{-1} & 4(a^3 + a + 1)(1+a)^{-3} \end{pmatrix} \\ H_{2k} &= \frac{1}{2a^2(a^2 + a + 1)} \begin{pmatrix} a^2(a^2 + 2a + 2) & 2a\rho^k \\ 2a\rho^k & 4(a^2 + 1)\rho^{2k} \end{pmatrix} \\ H_{2k+1} &= \frac{1}{2(1+a)^2(a^2 + a + 1)} \begin{pmatrix} (1+a)^2(2a^2 + 2a + 1) & -2a^2(1+a)\rho^k \\ -2a^2(1+a)\rho^k & 4(a^2 + 1)\rho^{2k} \end{pmatrix}. \end{aligned}$$

The step sizes are given, for $k > 0$, by

$$\alpha_0 = 1, \quad \alpha_1 = \frac{1}{a(1+a)}, \quad \alpha_{2k} = a\rho, \quad \alpha_{2k+1} = \frac{\rho}{a}.$$

Proof. A simple calculation verifies that the given sequence of iterates is indeed an exact-line-search BFGS sequence, by Proposition 4.1. Furthermore, since the function is strictly convex along each search direction, the given sequence is the unique exact-line-search BFGS sequence under the given initialization. The formulae for the inverse Hessian approximations are easy to verify directly by induction: see the Appendix. \square

Notice that the convergence rate ρ is unchanged under the transformation $a \leftarrow \frac{1}{a}$. This is not surprising, given the invariance of the method under scaling of the objective, and a consequent simple symmetry property.

In the example above, for very specific initial values, BFGS generates a sequence of points oscillating between two parabolas and converging linearly to the optimal solution, zero. We also observe, at each iteration, that the method crosses the axis on which the function is nonsmooth. Seemingly this property allows BFGS to “learn” the nonsmooth structure of the problem, coded into the inverse Hessian approximations. By contrast, as we see next, under general initial conditions, unless all the iterates except for the initial point lie on the two parabolas, the exact line search causes the simple nonsmooth BFGS method we consider here to halt at a nonsmooth point that is not optimal.

Proposition 4.3. *Consider the exact-line-search BFGS method applied to minimize the function*

$$f(u, v) = \max\{u^2 + v, u^2 - av\}$$

for some fixed parameter $a > 0$. Unless the first two iterates (u_0, v_0) and (u_1, v_1) satisfy the conditions $u_1 = (1 + a)^{-1}u_0$ and

$$v_1 = -\frac{2a}{a^2 + 3a + 1}u_1^2 \text{ or } v_1 = \frac{2}{a^2 + 3a + 1}u_1^2,$$

the algorithm will stop at a nonsmooth point after finitely many iterations.

Proof. For simplicity, we focus on the case $a = 1$. Assume the method generates an infinite sequence of smooth points $x_k = (u_k, v_k)^T$ for $k = 1, 2, 3, \dots$. We first claim that the coordinate v_k must change sign at every iteration. If not, then without loss of generality there exists an iteration n such that $v_{n-1} < 0$ and $v_n < 0$. The previous result ensures

$$(\nabla f_n - \nabla f_{n-1})^T(x_{n+1} - x_n) = 0,$$

so the search direction p_n must be in the direction of the vector $(0, 1)^T$. But the exact line search then causes termination at the nonsmooth point $x_{n+1} = (u_n, 0)^T$, contradicting our assumption.

Without loss of generality, we can next assume $v_{2k} > 0, v_{2k+1} < 0$ for all $k = 1, 2, 3, \dots$. By applying the previous result, we easily arrive at the recursion

$$\begin{aligned} u_{2k+1} &= -\frac{u_{2k} - u_{2k-1}}{2} \\ v_{2k+1} &= v_{2k} + \frac{(u_{2k} - u_{2k-1})(3u_{2k} - u_{2k-1})}{2}, \end{aligned}$$

and similarly

$$\begin{aligned} u_{2k} &= -\frac{u_{2k-1} - u_{2k-2}}{2}, \\ v_{2k} &= v_{2k-1} + \frac{(u_{2k-1} - u_{2k-2})(-3u_{2k-1} + u_{2k-2})}{2}. \end{aligned}$$

Hence we deduce

$$u_n = -\frac{u_{n-1} - u_{n-2}}{2}$$

for all iterates $n > 2$, and consequently

$$u_n + u_{n-1} = \frac{u_{n-1} + u_{n-2}}{2}.$$

Induction implies

$$u_n + u_{n-1} = \frac{u_1 + u_0}{2^{n-1}}.$$

We deduce, for $k = 1, 2, 3, \dots$,

$$\begin{aligned} u_{2k} &= \frac{2}{3} \left(\frac{1}{2^{2k}} - 1 \right) (u_1 + u_0) + u_0, \\ u_{2k+1} &= \frac{2}{3} \left(\frac{1}{2^{2k+1}} + 1 \right) (u_1 + u_0) - u_0. \end{aligned}$$

In particular, $u_{2k} \rightarrow \lambda := \frac{1}{3}u_0 - \frac{2}{3}u_1$ and $u_{2k+1} \rightarrow -\lambda$ as $k \rightarrow \infty$.

Now assume $\frac{u_1}{u_0} \neq \frac{1}{2}$, so $\lambda \neq 0$. Suppose first that $\lambda > 0$. (The case $\lambda < 0$ is similar.) Then for all large k we must have $u_{2k-1} < 0$ and $u_{2k} > 0$. By the previous result we know

$$(\nabla f_{2k} - \nabla f_{2k-1})^T (x_{2k+1} - x_{2k}) = 0,$$

so the search direction p_{2k} is in the direction $(-1, \mu)^T$ where $\mu = u_{2k} - u_{2k-1}$. By definition of the exact line search, we know $x_{2k+1} = x_{2k} + \beta(-1, \mu)^T$ where the scalar β minimizes

$$(u_{2k} - \beta)^2 + |v_{2k} + \beta\mu|.$$

If $v_{2k} + \beta\mu \geq 0$, then either $v_{2k+1} = 0$ or $v_{2k+1} > 0$. In the case $v_{2k+1} = 0$, our method stops at this nonsmooth point. If, on the other hand, $v_{2k+1} > 0$, then the same argument shows $v_{2k+2} = 0$.

Suppose, on the other hand, $v_{2k} + \beta\mu < 0$. Then, by its definition, β minimizes

$$(u_{2k} - \beta)^2 - (v_{2k} + \beta\mu),$$

so a quick calculation shows

$$\beta = \frac{3u_{2k} - u_{2k-1}}{4}.$$

Since $u_{2k} > 0$ and $u_{2k-1} < 0$, we deduce $\beta > 0$. We also know $v_{2k} > 0$ and $\mu > 0$, so we deduce

$$v_{2k+1} = v_{2k} + \beta\mu > 0,$$

which contradicts the property $v_{2k+1} < 0$.

Now consider the final case, where $\frac{u_1}{u_0} = \frac{1}{2}$ but $v_1 \neq -\frac{2}{5}u_1^2$. Then the formula above for the component u_n reduces to

$$u_n = \frac{u_0}{2^n},$$

and we can similarly deduce a formula for the component v_n :

$$v_{2k+1} = v_{2k} + \frac{(u_{2k} - u_{2k-1})(3u_{2k} - u_{2k-1})}{2} = v_{2k} - \frac{u_0^2}{2^{4k+1}}$$

and

$$v_{2k} = v_{2k-1} - \frac{(u_{2k-1} - u_{2k-2})(3u_{2k-1} - u_{2k-2})}{2} = v_{2k-1} + \frac{u_0^2}{2^{4k-1}}.$$

Hence v_n converges to $v_1 + \frac{2}{5}u_1^2$, which, by assumption, is nonzero. Thus v_n eventually does not change sign, which quickly gives a contradiction. \square

Corollary 4.1. *Consider the exact-line-search BFGS method applied to minimize the function*

$$f(u, v) = \max\{u^2 + v, u^2 - av\}$$

for some fixed parameter $a > 0$. Suppose the method generates an infinite sequence of smooth points x_0, x_1, x_2, \dots . Then that sequence must oscillate between the following two parabolas

$$v = \frac{2}{a^2 + 3a + 1}u^2 \quad \text{and} \quad v = -\frac{2a}{a^2 + 3a + 1}u^2,$$

converging linearly to the global minimizer zero.

Proof. Again we concentrate on the case $a = 1$ for simplicity. By the previous result, we can assume $v_{2k} > 0$, $v_{2k+1} < 0$, $u_0 = 2u_1$, and $v_1 = -\frac{2}{5}u_1^2$. Then we deduce

the formulae

$$u_{2k} = \frac{u_0}{2^{2k}}, \quad v_{2k} = \frac{2u_0^2}{5 \cdot 4^{2k}}, \quad u_{2k+1} = \frac{u_0}{2^{2k+1}}, \quad v_{2k+1} = -\frac{2u_0^2}{5 \cdot 4^{2k+2}},$$

so result follows. \square

To summarize this very simple theoretical case study, we observe two possible cases. Either the exact-line-search BFGS method converges linearly to the global minimizer zero, oscillating between two parabolas, or the line search causes the method to terminate prematurely at a nonoptimal nonsmooth point.

4.3 BFGS with inexact line search

We turn next from the idealized version of BFGS of the previous section to a more realistic version with an inexact line search. Again we focus on very simple examples, seeking insight on the method in the nonsmooth case, rather than extensive practical experience.

Recall that, for minimizing the function f , with the current iterate x_k and search direction p_k , the line search seeks a step length α_k : that is, a scalar t satisfying the two Wolfe conditions:

$$f(x_k + tp_k) \leq f(x_k) + c_1 t \nabla f(x_k)^T p_k \quad (4.3)$$

$$\nabla f(x_k + tp_k)^T p_k \geq c_2 \nabla f(x_k)^T p_k. \quad (4.4)$$

We here use the following algorithm [19, Alg. 2.6] to find a step length.

Algorithm 4.1. (*Inexact line search*)

```

 $\alpha \leftarrow 0;$ 
 $\beta \leftarrow +\infty;$ 
 $t \leftarrow 1;$ 
repeat
    if inequality (4.3) fails,  $\beta \leftarrow t$ 
    else if inequality (4.4) fails,  $\alpha \leftarrow t$ 
    else stop;
    if  $\beta < +\infty$ ,  $t \leftarrow (\alpha + \beta)/2$ 
    else  $t \leftarrow 2\alpha;$ 
end(repeat)

```

As we saw in the previous section, for nonsmooth examples, the behavior of BFGS with an exact line search can depend heavily on the initialization. By contrast, the behavior with an inexact line search in practice seems more robust. We consider here the question of to what extent we can gain insight on the convergence rate (*with respect to the number of function evaluations*) when using an inexact line search from the behavior with an exact line search.

The parametrized example: ill conditioning

For illustration, we return to the previous example

$$f(u, v) = u^2 + \max\{v, -av\}. \quad (4.5)$$

We are particularly interested in the broad dependence of the rate of convergence on the parameter a , which gives a certain measure of the “conditioning” of the problem. Notwithstanding the dependence of the standard smooth theory on the assumption $c_1 > 0$, for simplicity of exposition, we take $c_1 = 0$ and

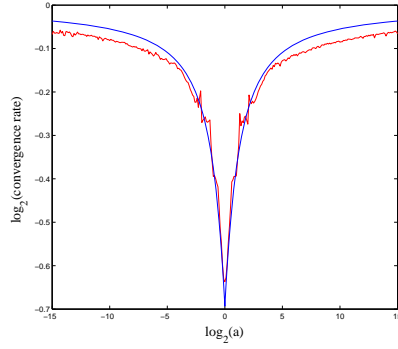


Figure 4.1: conjectured linear convergence rate versus observed convergence rate

$c_2 = 0.9$ here. Numerical experimentation shows that, with random initialization, inexact-line-search BFGS eventually crosses the v -axis at each iteration, and has a linear convergence rate, plotted in red on the figure. (This behavior is relatively insensitive to the choice of c_2 .)

A reasonable fit to the observed linear convergence rate is given by the function $r(a)$, defined for $0 < a < 1$ by

$$\log_2(r(a)) = \frac{\log_2(3a^2 + 3a + 1) - \log_2((a^2 + 3a + 3)(a + 1)^2)}{2 \log_2(1 + \frac{1}{a})}, \quad (4.6)$$

and for $a > 1$ by the obvious symmetry $r(a) = r(1/a)$. This function is plotted in blue on the figure. We arrive at this rough fit through the following loose intuition.

As Proposition 4.2 indicates, when applying exact-line-search BFGS to this function with appropriate starting points, we generate the iterates

$$x_{2k} = \left(\rho^k, \frac{2\rho^{2k}}{a^2 + 3a + 1} \right) \text{ and } x_{2k+1} = \left(\frac{\rho^k}{a + 1}, -\frac{2\rho^{2k+1}}{a^2 + 3a + 1} \right), \quad (4.7)$$

with step lengths

$$\alpha_{2k} = \left(1 + \frac{1}{a} \right)^{-2} \text{ and } \alpha_{2k+1} = (1 + a)^{-2}.$$

The linear convergence rate per iteration is, in this case,

$$\frac{f(x_{2k+1})}{f(x_{2k})} = \frac{1 + \frac{2a^2}{a^2+3a+1}}{1 + \frac{2}{a^2+3a+1}} \frac{1}{(a+1)^2} \quad \text{and} \quad \frac{f(x_{2k+2})}{f(x_{2k+1})} = \frac{1 + \frac{2}{a^2+3a+1}}{1 + \frac{2a^2}{a^2+3a+1}} \frac{a^2}{(1+a)^2}.$$

Consider the case when $a > 0$ is small. In that case, the odd iterations generate a large decrease in function value with a step length close to one. By contrast, the even iterations generate only a small decrease, and to do so need to use a small step length $(1 + 1/a)^{-2}$. We might expect a bisection-based line search to need roughly $\log_2((1 + 1/a)^2)$ function evaluations to locate the step. Hence an estimate of the convergence rate during those iterations is

$$r(a) = \left(\frac{f(x_{2k+1})}{f(x_{2k})} \right)^{\frac{1}{2 \log_2(1 + \frac{1}{a})}}. \quad (4.8)$$

Numerical experiments with inexact-line-search BFGS suggest that iterations analogous to the one above are in fact typical. Hence we arrive at the estimate (4.8), which does indeed give a reasonable fit to the experimental data. A similar argument applies to large $a > 0$.

We can explore this behavior in a more controlled fashion. Consider, for the moment, the behavior of our inexact line search when started at the exact-line-search iterates x_{2k} (or x_{2k+1}) described by (4.7) and searching in the corresponding directions p_{2k} (or p_{2k+1}). Numerical results (with $c_1 = 0, c_2 = 0.9$) suggest that the number of function evaluations needed by the line search only depends on a and doesn't depend on the iteration count k (except for its parity). The following lemma throws some light on that dependence.

Lemma 4.1. *Consider the function (4.5) with $a = 2^m - 1$ (for $m = 1, 2, 3, \dots$), and the exact-line-search BFGS iterates (4.7) and corresponding search directions p_{2k} and p_{2k+1} . With those iterates and search directions, the inexact line search would generate the step lengths $\alpha_{2k} = 1$ and $\alpha_{2k+1} = 2^{-m}$. On the other hand, in the case $a = 1/(2^m - 1)$, we generate $\alpha_{2k} = 2^{-m}$ and $\alpha_{2k+1} = 1$.*

Proof. We only prove the case when $a > 1$. The proof for $a < 1$ is similar.

For the even iterations, since $x_{2k} = (\rho^k, \frac{2\rho^{2k}}{a^2+3a+1})$ and $p_{2k} = (-\frac{(a+1)\rho^k}{a}, -\frac{2\rho^{2k}}{a^2})$, we deduce

$$\begin{aligned} f(x_{2k}) &= \left(1 + \frac{2}{a^2 + 3a + 1}\right)\rho^{2k}, \\ f(x_{2k} + p_{2k}) &= \left(\frac{3}{a^2} - \frac{2}{a^2 + 3a + 1}\right)\rho^{2k}, \\ \nabla f(x_{2k})^T p_{2k} &= -\left(2 + \frac{2}{a} + \frac{2}{a^2}\right)\rho^{2k}, \\ \nabla f(x_{2k} + p_{2k})^T p_{2k} &> 0. \end{aligned}$$

Since $\alpha = 1$ satisfies the Wolfe conditions, we deduce $\alpha_{2k} = 1$.

For the odd iterations, we have $x_{2k+1} = (\frac{\rho^k}{a+1}, -\frac{2\rho^{2k+1}}{a^2+3a+1})$ and $p_{2k+1} = (-\rho^k, 2\rho^{2k+1})$.

Then we obtain

$$\begin{aligned} f(x_{2k+1}) &= \left(\frac{1}{(a+1)^2} + \frac{2a\rho}{a^2 + 3a + 1}\right)\rho^{2k}, \\ x_{2k+1} + \alpha p_{2k+1} &= \left(\left(\frac{1}{a+1} - \alpha\right)\rho^k, \left(-\frac{2}{a^2 + 3a + 1} + 2\alpha\right)\rho^{2k+1}\right), \\ \nabla f(x_{2k+1})^T p_{2k+1} &= -\left(\frac{2}{a+1} + 2a\rho\right)\rho^{2k}. \end{aligned}$$

Consider the case $\alpha = 2^{-l}$ for some integer $l = 1, 2, 3, \dots, m-1$. We have

$$\nabla f(x_{2k+1} + \alpha p_{2k+1})^T p_{2k+1} = -\left(\frac{2}{a+1} - 2\alpha\right)\rho^{2k} + 2\rho^{2k+1} > 0.$$

Notice $\alpha \geq \frac{2}{a+1}$, so

$$\begin{aligned} f(x_{2k+1} + \alpha p_{2k+1}) &= \left(\alpha - \frac{1}{a+1}\right)^2 \rho^{2k} + \left(2\alpha - \frac{2}{a^2 + 3a + 1}\right)\rho^{2k+1} \\ &\geq \left(\frac{2}{a+1} - \frac{1}{a+1}\right)^2 \rho^{2k} + \left(\frac{2}{a+1} - \frac{2}{a^2 + 3a + 1}\right)\rho^{2k+1} \\ &> \frac{1}{(a+1)^2} \rho^{2k} + \frac{2a}{a^2 + 3a + 1} \rho^{2k+1} = f(x_{2k+1}). \end{aligned}$$

Therefore, the line search algorithm will successively try $\alpha = 1, \frac{1}{2}, \dots, \frac{1}{2^{m-1}}$, and finally $\alpha = \frac{1}{2^m} = \frac{1}{a+1}$. At this point we have

$$\nabla f(x_{2k+1} + \alpha p_{2k+1})^T p_{2k+1} = -\left(\frac{2}{a+1} - 2\alpha\right)\rho^{2k} + 2\rho^{2k+1} > 0,$$

and

$$\begin{aligned} f(x_{2k+1} + \alpha p_{2k+1}) &= \left(\alpha - \frac{1}{a+1}\right)^2 \rho^{2k} + \left(2\alpha - \frac{2}{a^2 + 3a + 1}\right) \rho^{2k+1} \\ &\leq \frac{2a}{(a+1)^2} \rho^{2k+1} \leq f(x_{2k+1}), \end{aligned}$$

so the Wolfe conditions are satisfied. The result follows. \square

The result above suggests that, if we were following the iterates generated by exact-line-search BFGS, then, for small $a > 0$, the “work” involved in each iteration, measured loosely by the number of function evaluations our *inexact* line search would take, is dominated by the even iterations, which depends on the factor $\log_2(1 + \frac{1}{a})$, a key ingredient of the estimate (4.8).

Example: a ridge

Rigorous analysis of inexact-line-search BFGS in the nonsmooth case seems very challenging in general. Here, for reassurance, we prove one very modest result. In the simplest possible case — a maximum of two affine functions (a “ridge”) — we can at least be sure that the method will not converge to a spurious limit. More precisely, we have the following result.

Proposition 4.4. *If the inexact-line-search BFGS method applied to the function $f(u, v) = |u| + v$ generates a sequence of iterates $x_k = (u_k, v_k)^T$ with $u_k \neq 0$ (for $k = 0, 1, 2, \dots$), then x_k does not converge.*

Proof. The Wolfe conditions hold at each iteration. Hence, if the current point satisfies $u_k > 0$, then the search direction $p_k = (m_k, l_k)^T$ satisfies $m_k < 0$, and at the next iteration we must have $u_{k+1} < 0$. A similar argument holds if $u_k < 0$.

We first prove $|m_k| > |l_k|$ for all iterations k . Without loss of generality, suppose $u_k > 0$. Since $\nabla f(x_k) = (1, 1)^T$, then

$$m_k + l_k = -(1, 1)H_k \begin{pmatrix} 1 \\ 1 \end{pmatrix} < 0.$$

Notice that $y_k = \nabla f(x_{k+1}) - \nabla f(x_k) = (-2, 0)^T$ and

$$V_k = I - (p_k^T y_k)^{-1} p_k y_k^T = \begin{pmatrix} 0 & 0 \\ -\frac{l_k}{m_k} & 1 \end{pmatrix}.$$

Hence

$$H_k = \begin{pmatrix} a_k & b_k \\ b_k & c_k \end{pmatrix} \Rightarrow H_{k+1} = V_k H_k V_k^T + \alpha_k (p_k^T y_k)^{-1} p_k p_k^T = \begin{pmatrix} -\frac{\alpha_k m_k}{2} & -\frac{\alpha_k l_k}{2} \\ -\frac{\alpha_k l_k}{2} & a_k (\frac{l_k}{m_k})^2 - 2b_k \frac{l_k}{m_k} + c_k - \alpha_k \frac{(l_k)^2}{2m_k} \end{pmatrix}.$$

Hence, $m_{k+1} = \frac{\alpha_k(l_k - m_k)}{2} > 0$, which implies $l_k - m_k > 0$. Combined with the fact that $m_k + l_k < 0$, we have $|m_k| > |l_k|$.

We now prove the proposition by contradiction. Suppose the sequence x_k converges. Then $\alpha_k m_k \rightarrow 0$ and $\alpha_k l_k \rightarrow 0$. Note

$$p_{k+1} = -H_{k+1} \nabla f(x_{k+1}) = - \begin{pmatrix} -\frac{\alpha_k m_k}{2} & -\frac{\alpha_k l_k}{2} \\ -\frac{\alpha_k l_k}{2} & c_{k+1} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha_k(l_k - m_k)}{2} \\ -\frac{\alpha_k l_k}{2} - c_{k+1} \end{pmatrix},$$

where $c_{k+1} = a_k (\frac{l_k}{m_k})^2 - 2b_k \frac{l_k}{m_k} + c_k - \alpha_k \frac{(l_k)^2}{2m_k}$. We deduce $m_{k+1} = \frac{\alpha_k(l_k - m_k)}{2} \rightarrow 0$. We now show that the positive number c_{k+1} stays bounded away from zero.

To this end, note by induction we have

$$H_{k+1} = V_k \cdots V_0 H_0 V_0^T \cdots V_k^T + \alpha_0 V_k \cdots V_1 (p_0^T y_0)^{-1} p_0 p_0^T V_1^T \cdots V_k^T + \cdots + \alpha_k (p_k^T y_k)^{-1} p_k p_k^T.$$

Since

$$V_i V_j = \begin{pmatrix} 0 & 0 \\ -\frac{l_i}{m_i} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\frac{l_j}{m_j} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{l_j}{m_j} & 1 \end{pmatrix} = V_j,$$

we simplify to

$$H_{k+1} = V_0 H_0 V_0^T + \alpha_0 V_1 (p_0^T y_0)^{-1} p_0 p_0^T V_1^T + \cdots + \alpha_k (p_k^T y_k)^{-1} p_k p_k^T.$$

It is easy to see that $p_k^T y_k > 0$ for all k . Hence the $(2, 2)$ -entry of the matrix H_k is increasing in k , and hence at least as large as the corresponding entry in the matrix $V_0 H_0 V_0^T$, namely

$$a_0 \left(\frac{l_0}{m_0} \right)^2 - 2b_0 \frac{l_0}{m_0} + c_0 > 0,$$

as required.

Finally, observe $l_{k+1} = -\frac{\alpha_k l_k}{2} - c_{k+1}$ cannot converge to zero, which contradicts the fact $|m_k| > |l_k|$. The result follows. \square

The idea of this proof extends to any maximum of two affine functions on \mathbf{R}^n . Note too how this example illustrates behavior that seems to drive the success of BFGS in the nonsmooth case: the inexact line search crosses the $u = 0$ axis (the manifold with respect to which the function is partly smooth) at each iteration, allowing the method to “learn” the nonsmooth structure.

4.4 A limit point with descent directions

In the above sections we illustrated good behavior of BFGS on some nonsmooth functions. In this section, we contrast with an illustration of some bad behavior.

The reference [19] conjectures that the inexact-line-search BFGS method converges to points that are *Clarke stationary*: for Lipschitz functions, this amounts to saying that we can find convex combinations of gradients at nearby points

that are arbitrarily small. For a large class of functions (for example, those of the form $h(c(\cdot))$ with h finite and convex and c smooth), Clarke stationarity guarantees that there exist no directions of linear descent. However, in general Clarke stationarity does not rule out descent directions: the function $x \mapsto -|x|$ at $x = 0$ is a simple example.

Here we show how BFGS can converge to a point at which there exist directions of linear descent. We begin with some relevant definitions (see [43]).

Definition 4.1. Consider a function $f : \mathbf{R}^n \rightarrow \overline{\mathbf{R}}$ and a point \bar{x} with $f(\bar{x})$ finite. Consider a vector $v \in \mathbf{R}^n$.

1. We call v a regular subgradient of f at \bar{x} , written $v \in \hat{\partial}f(\bar{x})$, if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \text{ as } x \rightarrow \bar{x};$$

2. We call \bar{x} regular stationary if $0 \in \hat{\partial}f(\bar{x})$. (In other words, \bar{x} is a local minimizer, up to first order.)
3. We call v a limiting subgradient of f at \bar{x} , written $v \in \partial f(\bar{x})$, if there are sequences $x^v \rightarrow \bar{x}$ with $f(x^v) \rightarrow f(\bar{x})$ and $v^v \in \hat{\partial}f(x^v)$ with $v^v \rightarrow v$.
4. We call \bar{x} limiting stationary if $0 \in \partial f(\bar{x})$.
5. If f is Lipschitz around \bar{x} and $0 \in \text{con}\{\partial f(\bar{x})\}$, then we call \bar{x} Clarke stationary.

A direction $p \in \mathbf{R}^n$ satisfying

$$\limsup_{t \downarrow 0} \frac{f(\bar{x} + tp) - f(\bar{x})}{t} < 0,$$

is called a direction of linear descent. (In this case, \bar{x} is clearly not regular stationary.)

Reference [18, Corollary 4.13] gives an example of exact-line-search BFGS applied to $f(x) = \|x\|$ in \mathbf{R}^2 . The complete statement is as follows.

Proposition 4.5. *Consider the exact-line-search BFGS method applied to the Euclidean norm in \mathbf{R}^2 , initialized by*

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}.$$

The method generates a sequence of vectors x_k that rotate clockwise through an angle of $\frac{\pi}{3}$ and shrink by a factor $\frac{1}{2}$ at each iteration.

In fact using our inexact line search (Algorithm 4.1) instead of the exact line search generates the same points, as the following calculation shows.

Proposition 4.6. *Consider inexact-line-search BFGS applied to the Euclidean norm in \mathbf{R}^2 . For any $0 < c_1 < \frac{2}{3}$ and $c_1 < c_2 < 1$, suppose the method is initialized by*

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } H_0 = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}.$$

Then the method generates a sequence of vectors x_k that rotate counterclockwise through an angle of $\frac{\pi}{3}$ and shrink by a factor $\frac{1}{2}$ at each iteration. Consider the matrix

$$R = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

Then in fact, at the k th iteration we have:

$$x_k = 2^{-k} R^{-k} x_0, \quad \alpha_k = \frac{1}{4}, \quad p_k = 2^{-k} R^{-k} \begin{pmatrix} -3 \\ \sqrt{3} \end{pmatrix} \text{ and } H_k = 2^{-k} R^{-k} H_0 R^k.$$

Proof. A direct calculation (see [19, Thm. 4.2]) shows that exact-line-search BFGS applied to the function $\|x\|$ initialized with x_0 and H_0 generates the sequence (x_k) . It is also easy to check that the exact step size for each iteration is $\frac{1}{4}$. In order to prove the result, it is sufficient to prove that the step size for inexact-line-search BFGS is also $\frac{1}{4}$ for each iteration.

Consider the k th iteration. The line search algorithm will try $t = 1$ first. Since $\nabla f(x_k)^T p_k = -3 \times 2^{-k}$ and $f(x_k + p_k) = \sqrt{7} \times 2^{-k}$, then

$$f(x_k + p_k) = 2^{-k} \sqrt{7} > 2^{-k}(1 - 3c_1) = f(x_k) + c_1 \nabla f(x_k)^T p_k$$

and

$$c_2 \nabla f(x_k)^T p_k = -3 \times 2^{-k} c_2 < \frac{9}{\sqrt{7}} \times 2^{-k} = \nabla f(x_k + p_k)^T p_k.$$

Hence the algorithm will try $t = \frac{1}{2}$. This time we note

$$f\left(x_k + \frac{1}{2}p_k\right) = 2^{-k} > 2^{-k}\left(1 - \frac{3}{2}c_1\right) = f(x_k) + \frac{c_1}{2} \nabla f(x_k)^T p_k,$$

and

$$c_2 \nabla f(x_k)^T p_k = -3 \times 2^{-k} c_2 < 3 \times 2^{-k} = \nabla f\left(x_k + \frac{1}{2}p_k\right)^T p_k.$$

Now the algorithm will try $t = \frac{1}{4}$. We observe

$$f\left(x_k + \frac{1}{4}p_k\right) = 2^{-(k+1)} < 2^{-k}\left(1 - \frac{3}{4}c_1\right) \left(c_1 < \frac{2}{3}\right) = f(x_k) + \frac{c_1}{4} \nabla f(x_k)^T p_k,$$

since $c_1 < 2/3$, and

$$c_2 \nabla f(x_k)^T p_k = -3 \times 2^{-k} c_2 < 0 = \nabla f\left(x_k + \frac{1}{4}p_k\right)^T p_k.$$

We deduce $\alpha_k = 1/4$. The result follows. \square

The above example indicates that inexact-line-search BFGS generates a sequence of points, which are located on the half-lines $\mathbf{R}_+(\cos \frac{n\pi}{3}, \sin \frac{n\pi}{3})$ (for integers n) and converge to zero. To construct an example where the algorithm converges to a point at which there exist descent directions, the idea is to ensure that BFGS still only visits those points, but to change the function values elsewhere.

Proposition 4.7. Consider inexact-line-search BFGS applied to the function

$$g(u, v) = \begin{cases} \sqrt{u^2 + v^2} \cdot \cos(18 \arctan \frac{v}{u}) & (u, v) \neq (0, 0) \\ 0 & (u, v) = (0, 0), \end{cases}$$

or equivalently in polar coordinates

$$g(r, \theta) = r \cos(18\theta).$$

For any $0 < c_1 < \frac{2}{3}$ and $c_1 < c_2 < 1$, if we initialize with

$$x_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } H_0 = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix},$$

the method generates the same sequence as in Proposition 4.6. Moreover, the method converges to the point zero, at which there exist directions of linear descent.

Proof. The existence of directions of linear descent at zero is clear, so we simply need to prove that the BFGS method generates the same sequence (x_k) as in Proposition 4.6 by induction.

Since

$$R = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos \frac{\pi}{3} & \sin \frac{\pi}{3} \\ -\sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix},$$

we compute

$$R^{-1} = \begin{pmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \quad R^k = \begin{pmatrix} \cos \frac{k\pi}{3} & \sin \frac{k\pi}{3} \\ -\sin \frac{k\pi}{3} & \cos \frac{k\pi}{3} \end{pmatrix} \quad R^{-k} = \begin{pmatrix} \cos \frac{k\pi}{3} & -\sin \frac{k\pi}{3} \\ \sin \frac{k\pi}{3} & \cos \frac{k\pi}{3} \end{pmatrix}.$$

Then $g(x_k) = g(2^{-k} R^{-k} x_0) = 2^{-k} = f(x_k)$. Furthermore we have

$$\left. \frac{\partial g(x)}{\partial u} \right|_{x=x_k} = \frac{u_k}{\|x_k\|} \cos\left(18 \arctan \frac{v_k}{u_k}\right) = \frac{u_k}{\|x_k\|} \quad \text{and} \quad \left. \frac{\partial g(x)}{\partial v} \right|_{x=x_k} = \frac{v_k}{\|x_k\|} \cos\left(18 \arctan \frac{v_k}{u_k}\right) = \frac{v_k}{\|x_k\|},$$

so

$$\nabla g(x_k) = \frac{x_k}{\|x_k\|} = \nabla f(x_k).$$

If we can show that, at each iteration k , the step size is always $\frac{1}{4}$, then the iterates indeed coincide, as required. The idea of the proof is to compare the functions $f = \|\cdot\|$ and g along the search directions at each iteration, and observe that the calculations during the inexact line search are identical. The figure illustrates.

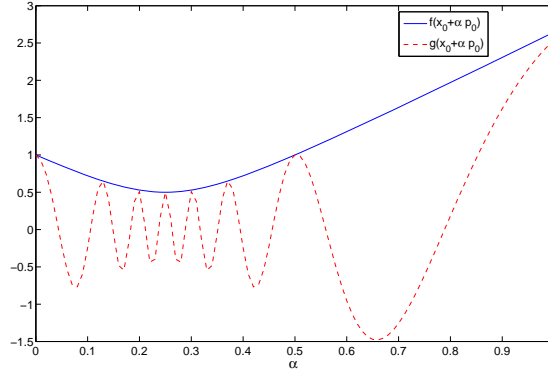


Figure 4.2: A comparison of f and g along search direction p_0 .

When $k = 0$, we have

$$p_0 = -H_0 \nabla g(x_0) = \begin{pmatrix} -3 \\ \sqrt{3} \end{pmatrix}, \quad \nabla g(x_0) = \nabla f(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider the line search algorithm applied to g . We first try $t = 1$. Since

$$g(x_0 + p_0) = g(-2, \sqrt{3}) = \sqrt{7} \cos(18 \arctan \frac{-\sqrt{3}}{2}) > 1 = g(x_0) > g(x_0) + c_1 \nabla g(x_0)^T p_0,$$

so $t = 1$ doesn't satisfy the first Wolfe condition. Moreover, as Figure 4.2 illustrates,

$$\nabla g(x_0 + p_0)^T p_0 \geq \nabla f(x_0 + p_0)^T p_0.$$

Since

$$\nabla f(x_0 + p_0)^T p_0 \geq c_2 \nabla f(x_0)^T p_0 = c_2 \nabla g(x_0)^T p_0,$$

the value $t = 1$ satisfies the second Wolfe condition for g . Therefore the line search algorithm will try $t = \frac{1}{2}$. As Figure 4.2 indicates,

$$g(x_0 + \frac{1}{2}p_0) = f(x_0 + \frac{1}{2}p_0) \text{ and } \nabla g(x_0 + \frac{1}{2}p_0) = \nabla f(x_0 + \frac{1}{2}p_0).$$

Hence $t = \frac{1}{2}$ doesn't satisfy the first Wolfe condition but does satisfy the second, following Proposition 4.6. Hence the line search will next try $t = \frac{1}{4}$. Since

$$g(x_0) = f(x_0), \nabla g(x_0) = \nabla f(x_0), g(x_0 + \frac{1}{4}p_0) = f(x_0 + \frac{1}{4}p_0), \nabla g(x_0 + \frac{1}{4}p_0) = \nabla f(x_0 + \frac{1}{4}p_0),$$

it follows that $t = \frac{1}{4}$ satisfies the Wolfe conditions. Hence the iterates coincide for $k = 1$.

We now proceed inductively, in similar fashion. We suppose that up to k th iteration the iterates coincide, and furthermore $p_k = 2^{-k}R^{-k}p_0$ and $H_k = R^{-k}H_0R^k$. We want to prove coincidence at the $k + 1$ th iteration. First notice that $x_k + tp_k = 2^{-k}R^{-k}(x_0 + tp_0)$ can be obtained by rotating $2^{-k}(x_0 + tp_0)$ counterclockwise through an angle of $\frac{k\pi}{3}$. Then we have $f(x_k + tp_k) = 2^{-k}f(x_0 + tp_0)$ and $g(x_k + tp_k) = 2^{-k}g(x_0 + tp_0)$. Therefore, by the above argument, the line search step size should be $\alpha_k = \frac{1}{4}$. As showed in Figure 4.3, there exist directions of linear descent at zero. Hence, the result follows. \square

In fact, a direct calculation shows that $\hat{\partial}g(0) = \emptyset$ and g is smooth on $\mathbf{R}^n / \{(0, 0)\}$, with $\|\nabla g\| \geq 1$ everywhere, so zero is not limiting stationary. However, zero is Clarke stationary.

4.5 Line-search BFGS versus trust-region BFGS

Given the apparent success of line-search BFGS methods on nonsmooth functions, it is natural to compare with trust-region versions. We consider here a

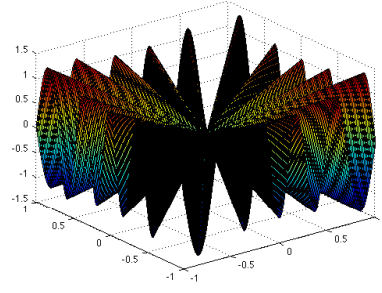


Figure 4.3: A picture of $g(u, v)$

trust-region BFGS algorithm from [32].

Algorithm 4.2. (*Trust-region BFGS algorithm*)

Given a starting point x_0 , initial Hessian approximation B_0 , trust-region radius Δ_0 , maximum number of iteration N , parameters $\eta \in (0, 10^{-3})$ and $r \in (0, 1)$;

$k \leftarrow 0$;

while $k < N$;

Exactly solve the subproblem

$$s_k \leftarrow \operatorname{argmin} \left\{ \nabla f_k^T s + \frac{1}{2} s^T B_k s : \|s\| \leq \Delta_k \right\};$$

Compute

$$y_k \leftarrow \nabla f(x_k + s_k) - \nabla f_k$$

$$ared \leftarrow f_k - f(x_k + s_k)$$

$$pred \leftarrow -(\nabla f_k^T s_k + \frac{1}{2} s_k^T B_k s_k);$$

if $\frac{ared}{pred} > \eta$

$$x_{k+1} \leftarrow x_k + s_k;$$

else $x_k = x_{k+1}$;

end(if)

```

if  $\frac{ared}{pred} > 0.75$ 
    if  $\|s_k\| \leq 0.8\Delta_k$ ,  $\Delta_{k+1} \leftarrow \Delta_k$ ;
    else  $\Delta_{k+1} \leftarrow 2\Delta_k$ ;
    end(if)
elseif  $0.1 \leq \frac{ared}{pred} \leq 0.75$ ,  $\Delta_{k+1} \leftarrow \Delta_k$ ;
else  $\Delta_{k+1} = 0.5\Delta_k$ ;
end(if)
if  $|s_k^T(y_k - B_k s_k)| \geq r\|s_k\| \cdot \|y_k - B_k s_k\|$ ,
    
$$B_{k+1} \leftarrow B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k};$$

    else  $B_{k+1} \leftarrow B_k$ ;
    end(if)
     $k \leftarrow k + 1$ ;
end(while)

```

Numerical experiments show that line-search BFGS methods work well for broad classes of nonsmooth functions, while trust-region versions fail even on simple examples. In this section, we use the simple nonsmooth function $f(u, v) = u^2 + |v|$ to explore some intuitive reasons for the success of line-search BFGS methods over their trust-region counterparts.

We present some simple numerical experiments. The following graph on the left is an example where trust-region BFGS fails to converge to the optimal solution. In contrast, the right graph shows the success of line-search BFGS on the same example.

Points on the axis $v = 0$ are nonsmooth. Numerical results show that the line-search BFGS method generates a sequence of points that eventually cross that

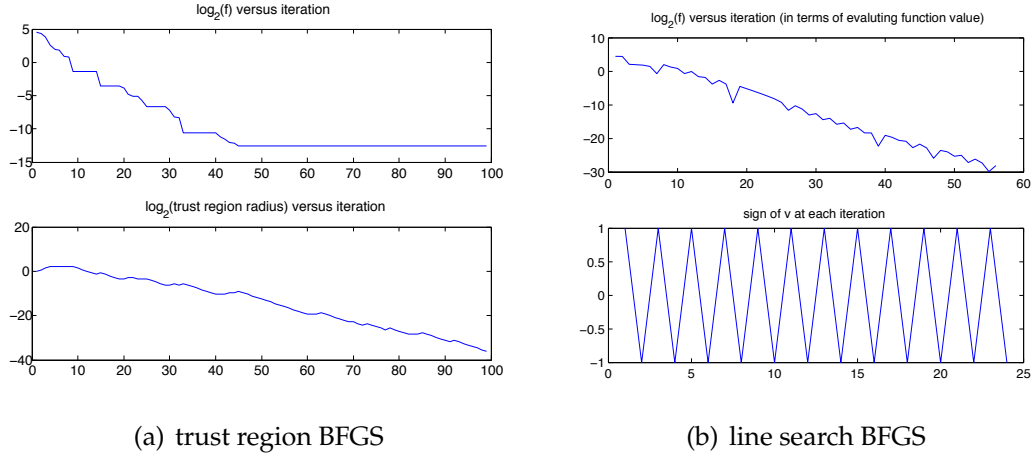


Figure 4.4: numerical results on $f(u, v) = u^2 + |v|$.

axis at every iteration (see the lower right figure above). Indeed, this property can be proved analytically for exact-line search BFGS, as we saw above. However, trust-region BFGS method seems to satisfy no analogous property. The trust region seems overly restrictive on the updated point and approximate Hessian. Somehow, the line-search BFGS method seems to detect the nonsmooth structure of the function better than the trust-region BFGS method.

Secondly, the line-search BFGS method updates the approximate Hessian when it finds a point satisfying the Wolfe conditions along the current search direction, and the Wolfe conditions seem to ensure that the updated point is satisfactory for this update. However, the trust-region BFGS updates the approximated Hessian matrix at each iteration, even when the current subproblem is not a good approximation of the original problem around the current point.

Thirdly, numerical results show that the radius of the trust region converges to zero quickly (see the lower left figure above). When the trust region is small, the method cannot take a big step even though the subproblem is a good approximation of the original problem. This causes the method to converge very

slowly. In addition, for the same reason, the method fails to take advantage of a well approximated subproblem to better update the approximate Hessian.

4.6 Conclusion

This chapter presents some initial explorations of the line-search BFGS method on some very simple nonsmooth examples. The examples provide some interesting illustrations of how the method seems gradually to identify the nonsmooth structure. In particular, the inexact line-search algorithm seems important in responding to nonsmoothness in the objective function. However, the theory underlying this phenomenon is poorly understood.

CHAPTER 5

FURTHER DEVELOPMENT AND QUESTIONS

5.1 Introduction

In this section, we will first discuss some further research results based on Chapter 3. After that, we will list several open questions, which can serve as a stimulus for future research.

5.2 Calculus rules for generalized Hessian mappings

In Chapter 3, we computed the generalized Hessian mappings for partly smooth and prox-regular functions. By combining those results with calculus rules for partial smoothness and prox-regularity, we can derive corresponding calculus rules for generalized Hessian mappings.

The paper [17] proved a variety of calculus rules for partly smooth functions. The following are examples.

Theorem 5.1. *Given an open set $W \subset \mathbf{R}^n$ containing a point z_0 , a smooth map $\Phi : W \rightarrow \mathbf{R}^m$, and a manifold $\mathcal{M} \subset \mathbf{R}^m$, suppose Φ satisfies*

$$R(\nabla\Phi(z_0)) + T_{\mathcal{M}}(\Phi(z_0)) = \mathbf{R}^m.$$

If the function $h : \mathbf{R}^m \rightarrow \overline{\mathbf{R}}$ is partly smooth at $\Phi(z_0)$ relative to \mathcal{M} , then the composition $h \circ \Phi$ is partly smooth at z_0 relative to $\Phi^{-1}(\mathcal{M})$.

Proof. See [17, Thm. 4.2]. □

Theorem 5.2. Consider manifolds $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ in \mathbf{R}^n . Suppose the function $h_i : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is partly smooth at the point z_0 relative to \mathcal{M}_i for each i . Assume furthermore the condition

$$\sum_i y_i = 0 \text{ and } y_i \in N_{\mathcal{M}_i}(z_0) \text{ for each } i \Rightarrow y_i = 0 \text{ for each } i.$$

Then the function $\sum_i h_i$ is partly smooth at z_0 relative to $\cap_i \mathcal{M}_i$.

Proof. See [17, Cor. 4.6]. □

A recent paper [39] showed that, under certain assumptions, the prox-regularity property is also preserved under the operations of composition and addition.

Theorem 5.3. Let $f_i, i = 1, 2$ be extended real-valued functions on \mathbf{R}^n . Consider $\bar{x} \in [\text{dom} f_1 \cap \text{dom} f_2]$, and assume that

$$\text{the only choice of } v_i \in \partial^\infty f_i(\bar{x}) \text{ with } v_1 + v_2 = 0 \text{ is } v_1 = v_2 = 0.$$

Let $\bar{v} \in \partial f(\bar{x})$, where $f(x) = f_1(x) + f_2(x)$. Assume further that for each $v_i \in \partial f_i(\bar{x})$ with $v_1 + v_2 = \bar{v}$, the function f_i is prox-regular at \bar{x} for v_i . Then f is prox-regular at \bar{x} for \bar{v} and there exists $\epsilon > 0$ such that

$$[\partial f_1(x) + \partial f_2(x)] \cap B(\bar{v}, \epsilon) = \partial f(x) \cap B(\bar{v}, \epsilon)$$

whenever $|x - \bar{x}| < \epsilon$ with $|f(x) - f(\bar{x})| < \epsilon$.

Proof. See [39, Thm. 2.2]. □

Theorem 5.4. Let $f(x) = g(F(x))$, where $F : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is of class C^2 , $g : \mathbf{R}^m \rightarrow \bar{\mathbf{R}}$ is lower semicontinuous and proper, and suppose that the following property is satisfied

$$y \in \partial^\infty g(F(\bar{x})) \text{ and } \nabla F(\bar{x})^* y = 0 \Rightarrow y = 0.$$

Assume further that $\bar{v} \in \partial f(\bar{x})$ is a vector such that the function g is prox-regular at $F(\bar{x})$ for every $y \in \partial g(F(\bar{x}))$ with $\nabla F(\bar{x})^* y = \bar{v}$. Then f is prox-regular at \bar{x} for \bar{v} .

Proof. See [39, Thm. 2.1]. □

Now we are ready to prove calculus rules for generalized Hessian mappings.

Theorem 5.5. *Let $f_i : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ be partly smooth at \bar{x} relative to \mathcal{M}_i for $i = 1, 2$. Assume furthermore the condition*

$$y_1 + y_2 = 0 \text{ and } y_i \in N_{\mathcal{M}_i}(\bar{x}) \Rightarrow y_i = 0 \text{ for each } i.$$

Suppose that f_i is prox-regular and subdifferentially continuous at \bar{x} for $v_i \in \text{ri } \partial f_i(\bar{x})$. If $f(x) := f_1(x) + f_2(x)$ is also subdifferentially continuous at \bar{x} for $\bar{v} := v_1 + v_2$, then

$$\partial^2 f(\bar{x}|\bar{v}) = \partial^2 f_1(\bar{x}|v_1) + \partial^2 f_2(\bar{x}|v_2).$$

Proof. Theorem 5.2 implies that $f_1 + f_2$ is partial smooth at \bar{x} relative to $\mathcal{M}_1 \cap \mathcal{M}_2$. Since $\partial^\infty f_i(\bar{x}) \subset N_{\mathcal{M}_i}(\bar{x})$, then the only choice of $y_i \in \partial^\infty f_i(\bar{x})$ with $y_1 + y_2 = 0$ is $y_1 = y_2 = 0$. Therefore, we have $f_1 + f_2$ is prox-regular at \bar{x} for \bar{v} by Theorem 5.3.

First notice that, the only choice of $u_1 + u_2 = \bar{v}$ with $u_i \in \partial f_i(\bar{x})$ is $u_i = v_i$. Suppose, for each $i = 1, 2$, we have $\mathcal{M}_i = \{x \in \mathbf{R}^n \mid \Phi_j^i(x) = 0, j = 1, \dots, m_i\}$ locally around \bar{x} . Then $\mathcal{M}_1 \cap \mathcal{M}_2 = \{x \in \mathbf{R}^n \mid \Phi_j^i(x) = 0, j = 1, \dots, m_i, i = 1, 2\}$ locally around \bar{x} . Hence we have

$$N_{\mathcal{M}_1 \cap \mathcal{M}_2}(\bar{x}) = N_{\mathcal{M}_1}(\bar{x}) + N_{\mathcal{M}_2}(\bar{x}), \quad T_{\mathcal{M}_1 \cap \mathcal{M}_2}(\bar{x}) = T_{\mathcal{M}_1}(\bar{x}) \cap T_{\mathcal{M}_2}(\bar{x}).$$

Let h^i be a smooth representative of f_i around \bar{x} and $L_i(x) = h^i(x) + \sum_j^{m_i} \bar{\lambda}_j^i \Phi_j^i(x)$ with $\nabla L_i(\bar{x}) = v_i$.

Now Theorem 3.6 gives

$$\partial^2 f(\bar{x}|\bar{v})(w) = \begin{cases} \nabla^2(L_1 + L_2)(\bar{x}) + N_{\mathcal{M}_1 \cap \mathcal{M}_2}(\bar{x}) & w \in T_{\mathcal{M}_1 \cap \mathcal{M}_2}(\bar{x}) \\ \emptyset & \text{otherwise,} \end{cases}$$

which implies that $\partial^2 f(\bar{x}|\bar{v}) = \partial^2 f_1(\bar{x}|v_1) + \partial^2 f_2(\bar{x}|v_2)$. Hence, the theorem follows. \square

Theorem 5.6. *Let $f = g \circ F$, where $F : \mathbf{R}^m \rightarrow \mathbf{R}^n$ is C^2 -smooth, $g : \mathbf{R}^n \rightarrow \bar{\mathbf{R}}$ is lower semi-continuous and proper. Suppose g is partly smooth at $F(\bar{x})$ relative to $\mathcal{M} \subset \mathbf{R}^n$, and moreover, the following constraint qualification is satisfied*

$$R(\nabla F(\bar{x})) + T_{\mathcal{M}}(F(\bar{x})) = \mathbf{R}^n.$$

Suppose that g is prox-regular and subdifferentially continuous at $F(\bar{x})$ for $\bar{y} \in \text{ri } \partial g(F(\bar{x}))$. If $g \circ F$ is also subdifferentially continuous at \bar{x} for $\bar{v} := \nabla F(\bar{x})^ \bar{y}$, then we have*

$$\partial^2 (g \circ F)(\bar{x}|\bar{v})(w) = \begin{cases} \nabla^2 \langle F, \bar{y} \rangle w + \nabla F(\bar{x})^* \partial^2 g(F(\bar{x})|\bar{y})(\nabla F(\bar{x})w) & w \in T_{F^{-1}(\mathcal{M})}(\bar{x}) \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. First notice that $R(\nabla F(\bar{x})) + T_{\mathcal{M}}(F(\bar{x})) = \mathbf{R}^n$ implies the only choice of $y \in \partial^\infty g(F(\bar{x}))$ and $\nabla F(\bar{x})^* y = 0$ is $y = 0$, and \bar{y} is the unique solution of $\nabla F(\bar{x})^* \bar{y} = \nabla F(\bar{x})^* y$ with $y \in \partial g(F(\bar{x}))$. Then Theorems 5.1 and 5.4 show that $g \circ F$ is partly smooth at \bar{x} relative to $F^{-1}(\mathcal{M})$, and $g \circ F$ is prox-regular at \bar{x} for \bar{v} . Since $\bar{y} \in \text{ri } \partial g(F(\bar{x}))$ and $\partial(g \circ F)(\bar{x}) = \nabla F(\bar{x})^* \partial g(F(\bar{x}))$, then $\bar{v} = \nabla F(\bar{x})^* \bar{y} \in \text{ri } \partial(g \circ F)(\bar{x})$.

Let h be a smooth representative of g around $F(\bar{x})$ and Φ_j be local equations for \mathcal{M} . Then $\Phi_j \circ F$ are local equations for $F^{-1}(\mathcal{M})$ with $N_{F^{-1}(\mathcal{M})}(\bar{x}) = \nabla F(\bar{x})^* N_{\mathcal{M}}(F(\bar{x}))$ by the constraint qualification. Let

$$L(x) = (h \circ F)(x) + \sum_j \bar{\lambda}_j (\Phi_j \circ F)(x)$$

with $\nabla L(\bar{x}) = \bar{v}$. According to Theorem 3.6, we have

$$\partial^2(g \circ F)(\bar{x}|\bar{v})(w) = \begin{cases} \nabla^2 L(\bar{x})w + N_{F^{-1}(\mathcal{M})}(\bar{x}) & (w \in T_{F^{-1}(\mathcal{M})}(\bar{x})) \\ \emptyset & (\text{otherwise}). \end{cases}$$

Since

$$\begin{aligned} & \nabla^2 L(\bar{x})w + N_{F^{-1}(\mathcal{M})}(\bar{x}) \\ = & \nabla^2 \langle F, \bar{y} \rangle w + \nabla F(\bar{x})^* \left(\nabla^2 h(F(\bar{x})) + \sum_j \bar{\lambda}_j \nabla^2 \Phi_j(F(\bar{x})) \right) (\nabla F(\bar{x})w) + \nabla F(\bar{x})^* N_{\mathcal{M}}(F(\bar{x})) \\ = & \nabla^2 \langle F, \bar{y} \rangle w + \nabla F(\bar{x})^* \partial^2 g(F(\bar{x})|\bar{y})(\nabla F(\bar{x})w), \end{aligned}$$

then the theorem follows. \square

Note that [30] also talks about second-order subdifferential chain rules: [30, Thm. 3.1] gives an exact chain rule under a full rank condition, while [30, Thm. 3.2] shows inclusion-type chain rules for strongly amenable compositions. By concentrating on partly smooth functions, we get an exact chain rule and the proof is simpler.

5.3 Second-order conditions for a nonsmooth example

In recent years, a lot of research articles have studied the regularized minimization problems with nonconvex, nonsmooth, or non-Lipschitz penalty functions. The reason is that extensive applications of these problems have been found in signal processing, image restoration, and variable selection.

The paper [5] studies the optimality conditions for a special interesting class of regularized minimization problems:

$$\min_{x \in \mathbf{R}^n} h(x) := \theta(x) + \lambda \sum_{i=1}^m \varphi(|d_i^T x|),$$

where θ is twice differentiable, $\lambda > 0$, and the penalty function φ satisfies the following assumption.

Assumption 5.1.

- 1 φ is differentiable and φ' is locally Lipschitz continuous in $(0, \infty)$;
- 2 φ is continuous at 0 with $\varphi(0) = 0$, $\varphi'(0^+) = +\infty$ and $\varphi'(t) \geq 0$ for all $t > 0$.

For any nonzero $\bar{x} \in \mathbf{R}^n$, let

$$I_{\bar{x}} = \{ i \mid d_i^T \bar{x} = 0 \} \text{ and } J_{\bar{x}} = \{ i \mid d_i^T \bar{x} \neq 0 \}.$$

Let $Y_{\bar{x}}$ be an $n \times (n - l)$ matrix whose columns form an orthonormal basis for the span of $\{d_i \mid i \in I_{\bar{x}}\}$ and $Z_{\bar{x}}$ be an $n \times l$ matrix whose columns are an orthonormal basis for the corresponding orthogonal complement. Note that $\bar{x} \neq 0$ implies that $Z_{\bar{x}}$ is nontrivial.

In this section, we illustrate our results in Chapter 3 by deriving the second-order sufficient conditions for the above example, which turn out to be equivalent to the second-order conditions in [5, Thm. 2.4]. In order to apply our results, we need a slightly stronger assumption

Assumption 5.2.

1. φ is C^2 smooth in $(0, \infty)$;
2. φ is continuous at 0 with $\varphi(0) = 0$, $\varphi'(0^+) = +\infty$.

Under this assumption, [5, Thm. 2.4] can be reduced to the following statement.

Theorem 5.7. Suppose Assumption 5.2 holds, and consider a nonzero point $\bar{x} \in \mathbf{R}^n$. Define a function $w(x) := \theta(x) + \lambda \sum_{i \in J_{\bar{x}}} \varphi(|d_i^T x|)$. Assume that

$$Z_{\bar{x}}^T \nabla w(\bar{x}) = 0 \text{ and } Z_{\bar{x}}^T \nabla^2 w(\bar{x}) Z_{\bar{x}} \text{ is positive definite,}$$

where $\nabla^2 w(\bar{x})$ is given by $\nabla^2 \theta(\bar{x}) + \lambda \sum_{i \in J_{\bar{x}}} d_i d_i^T \varphi''(|d_i^T \bar{x}|)$. Then \bar{x} is a strict local minimizer of the function h .

Proof. See [5, Thm. 2.4]. □

Let \mathcal{M} be the linear space spanned by the columns of $Z_{\bar{x}}$:

$$\mathcal{M} = \{x \mid d_i^T x = 0, i \in I_{\bar{x}}\}.$$

Obviously, $N_{\mathcal{M}} = \{ \sum c_i d_i \mid i \in I_{\bar{x}}, c_i \in \mathbf{R} \}$ and $T_{\mathcal{M}}(\bar{x}) = \mathcal{M}$. Without essential loss of generality, suppose $I_{\bar{x}} \neq \emptyset$. (Otherwise the result is easy.)

Lemma 5.1. Under Assumption 5.2, for any $x \in \mathcal{M}$, the function h is regular at x with

$$\partial h(\bar{x}) = \left\{ \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\}$$

and

$$\partial^\infty h(\bar{x}) = \left\{ \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\}.$$

Proof. For any x close to \bar{x} , we have $d_i^T x \neq 0$ for $i \in J_{\bar{x}}$. Therefore, the function $w(x)$ is C^2 smooth near \bar{x} . For any $v = \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i$ and x close to \bar{x} , we have

$$\begin{aligned} & h(x) - h(\bar{x}) - \langle v, x - \bar{x} \rangle \\ &= \sum_{i \in I_{\bar{x}}} (\lambda \varphi(|d_i^T x|) - \langle c_i d_i, x - \bar{x} \rangle) + w(x) - w(\bar{x}) - \langle \nabla w(\bar{x}), x - \bar{x} \rangle \\ &\geq \sum_{i \in I_{\bar{x}}} (\lambda \varphi(|d_i^T x|) - \langle c_i d_i, x \rangle) + o(|x - \bar{x}|) \\ &\geq o(|x - \bar{x}|) \text{ (since } \varphi'(0^+) = +\infty). \end{aligned}$$

Therefore $\nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i \in \hat{\partial} h(\bar{x})$ for any $c_i \in \mathbf{R}$. Moreover, any $v \in \hat{\partial} h(\bar{x})$ should have this form. If not, then there exists $q \in T_{\mathcal{M}}(\bar{x}) = \mathcal{M}$ such that $v = \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i + q$. Since $v \in \hat{\partial} h(|d_i^T \bar{x}|)$, then we have

$$\begin{aligned} w(x) + \lambda \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T x|) &\geq w(\bar{x}) + \langle v, x - \bar{x} \rangle + o(|x - \bar{x}|) \\ \Rightarrow \lambda \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T x|) &\geq \langle \sum_{i \in I_{\bar{x}}} c_i d_i + q, x - \bar{x} \rangle - w(x) + w(\bar{x}) + \langle \nabla w(\bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|) \\ \Rightarrow \lambda \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T x|) &\geq \langle \sum_{i \in I_{\bar{x}}} c_i d_i + q, x - \bar{x} \rangle + o(|x - \bar{x}|). \end{aligned}$$

Let $x = \epsilon q + \bar{x}$. Then we have

$$\begin{aligned} \sum_{i \in I_{\bar{x}}} \varphi(|d_i^T (\epsilon q + \bar{x})|) &\geq \langle \sum_{i \in I_{\bar{x}}} c_i d_i + q, \epsilon q \rangle + o(|x - \bar{x}|) \\ \Rightarrow 0 &\geq \epsilon \|q\|^2 + o(\epsilon \|q\|). \end{aligned}$$

We arrive at a contradiction. Therefore

$$\hat{\partial} h(\bar{x}) = \left\{ \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\}.$$

By the same argument, for any x close to \bar{x} we can derive that

$$\hat{\partial} h(x) = \left\{ \nabla w(x) + \sum_{\{i \in I_{\bar{x}}, d_i^T x \neq 0\}} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i + \sum_{\{i \in I_{\bar{x}}, d_i^T x = 0\}} c_i d_i \mid c_i \in \mathbf{R} \right\}.$$

Hence, $\partial h(\bar{x}) = \hat{\partial} h(\bar{x})$. It is not hard to derive

$$\partial^\infty h(\bar{x}) = \left\{ \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\}$$

by the definition of horizon subgradients. Moreover, we have for some c_i

$$(\partial h(\bar{x}))^\infty = \left\{ \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\} = \partial^\infty h(\bar{x}).$$

Therefore, h is regular at \bar{x} according to Corollary 2.1. Using the same argument, we can prove that h is regular at any point near \bar{x} . \square

Lemma 5.2. *Under Assumption 5.2, given any $M > 0$, there exists an $\epsilon > 0$ such that $|v| > M$ for any $v \in \partial h(x)$ with $x \notin \mathcal{M}$ and $|x - \bar{x}| \leq \epsilon$.*

Proof. There exists $\delta > 0$ such that $w(x)$ is C^2 smooth for any $|x - \bar{x}| \leq \delta$. Let $B_\delta(\bar{x}) = \{x \mid |x - \bar{x}| \leq \delta\}$. For $x \in B_\delta(\bar{x})$ and $x \notin \mathcal{M}$, we have that

$$\partial h(x) = \left\{ \nabla w(x) + \sum_{\{i \in I_{\bar{x}}: d_i^T x \neq 0\}} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i + \sum_{\{i \in I_{\bar{x}}: d_i^T x = 0\}} c_i d_i \mid c_i \in \mathbf{R} \right\}.$$

For such x , we can divide the set $I_{\bar{x}}$ into three disjoint sets E_x, L_x and G_x satisfying

1. $i \in E_x$ when $d_i^T x = 0$; $i \in L_x$ when $d_i^T x < 0$; $i \in G_x$ when $d_i^T x > 0$;
2. $I_{\bar{x}} = E_x \cup L_x \cup G_x$ and $L_x \cup G_x \neq \emptyset$.

There are a finite number of such combinations, which we can denote by E_k, L_k, G_k with $k \in \{1, \dots, K\}$. We define set S_k by

$$S_k = \left\{ x \mid d_i^T x = 0 \ (i \in E_k); d_i^T x < 0 \ (i \in L_k); d_i^T x > 0 \ (i \in G_k) \right\} \cap B_\delta(\bar{x}).$$

Given $y \in S_k$, let $d = \frac{y - \bar{x}}{|y - \bar{x}|}$ and $y = \bar{x} + |y - \bar{x}|d$. Then for any $v \in \partial h(y)$, we have

$$\begin{aligned} v^T d &= \left(\nabla w(y) + \sum_{i \in L_k} \text{sign}(d_i^T y) \varphi'(|d_i^T y|) \lambda d_i + \sum_{i \in G_k} \text{sign}(d_i^T y) \varphi'(|d_i^T y|) \lambda d_i + \sum_{i \in E_k} c_i d_i \right)^T d \\ &= \nabla w(y)^T d + \sum_{i \in L_k} \text{sign}(d_i^T y) \varphi'(|d_i^T y|) \lambda d_i^T d + \sum_{i \in G_k} \text{sign}(d_i^T y) \varphi'(|d_i^T y|) \lambda d_i^T d. \end{aligned}$$

Since $d_i^T y$ and $d_i^T d$ have the same sign, then

$$v^T d = \nabla w(y)^T d + \sum_{i \in L_k \cup G_k} \varphi'(|y - \bar{x}| \cdot |d_i^T d|) \lambda |d_i^T d|.$$

Since $\varphi'(0+) = +\infty$, there exists $\delta_k > 0$ such that $v^T d > M$ as long as $|y - \bar{x}| \cdot |d_i^T d| \leq \delta_k$ for any $i \in L_k \cup G_k$. Let $\epsilon_k \leq \frac{\delta_k}{\max_{i \in L_k \cup G_k} |d_i|}$. Then we have $v^T d > M$. Since d is a unit

vector, then $|v| > M$. For any $x \in S_k$ with $|x - \bar{x}| < \epsilon_k$ and $v \in \partial h(x)$, we have that

$$\begin{aligned} v^T d &= \left(\nabla w(x) + \sum_{i \in L_k} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i + \sum_{i \in G_k} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i + \sum_{i \in E_k} c_i d_i \right)^T d \\ &= \nabla w(x)^T d + \sum_{i \in L_k} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i^T d + \sum_{i \in G_k} \text{sign}(d_i^T x) \varphi'(|d_i^T x|) \lambda d_i^T d. \end{aligned}$$

Since $d_i^T x, d_i^T y, d_i^T d$ have the same sign, then

$$v^T d = \nabla w(x)^T d + \sum_{i \in L_k \cup G_k} \varphi'(|d_i^T x|) \lambda |d_i^T d|.$$

From $|d_i^T x| \leq |d_i^T (x - \bar{x})| \leq |d_i| \cdot |x - \bar{x}| \leq \delta_k$, we have that $v^T d > M$ for any $v \in \partial h(x)$. Therefore, for any $x \in S_k$ with $|x - \bar{x}| \leq \epsilon_k$, we have $|v| > M$. Let $\epsilon = \min_k \epsilon_k$. For any $|x - \bar{x}| \leq \epsilon$ and $x \notin \mathcal{M}$, there exists k such that $x \in S_k$, and hence $|v| > M$ for any $v \in \partial h(x)$. \square

The above lemma indicates that given any $v \in \partial h(\bar{x})$, there is no sequence of $v_n \in \partial h(x_n)$ with $x_n \notin \mathcal{M} \rightarrow \bar{x}$ such that $v_n \rightarrow v$. Now we can prove the following proposition.

Proposition 5.1. *Consider a nonzero vector $\bar{x} \in \mathbf{R}^n$. Under Assumption 5.2, we have*

1. *$h(x)$ is partly smooth at \bar{x} relative to \mathcal{M} ;*
2. *Furthermore, h is also prox-regular at \bar{x} .*

Proof. First note that $T_{\mathcal{M}}(\bar{x}) = \mathcal{M}$ and $N_{\mathcal{M}}(\bar{x})$ is the linear space spanned by d_i with $i \in I_{\bar{x}}$. According to Lemma 5.1, the function h is regular at any point x in \mathcal{M} . Since $N_{\mathcal{M}}$ is the linear space spanned by d_i with $i \in I_{\bar{x}}$, then $N_{\mathcal{M}}(\bar{x}) = \text{par } \partial h(\bar{x})$. For any $x \in \mathcal{M}$, we also have $\partial h(x) = \{\nabla w(x) + \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R}\}$, which implies that $\partial h(\bar{x})$ is continuous at \bar{x} relative to \mathcal{M} . Hence, $h(x)$ is partly smooth at \bar{x} relative to \mathcal{M} .

Next, we are going to prove that h is prox-regular at \bar{x} . It is sufficient to prove that for any $\bar{v} \in \partial h(\bar{x})$, there exist $\epsilon > 0$ and $\rho > 0$ such that

$$h(x') \geq h(x) + \langle v, x' - x \rangle - \frac{\rho}{2} |x - x'|^2$$

providing $|x - \bar{x}| < \epsilon$, $|x' - \bar{x}| < \epsilon$, and $|v - \bar{v}| < \epsilon$. Assuming ϵ is small enough and $|v - \bar{v}| < \epsilon$, we can deduce that x has to lie in \mathcal{M} by Lemma 5.2. Therefore we only need to prove that

$$w(x') - w(x) - \langle \nabla w(x), x' - x \rangle + \sum_{i \in I_{\bar{x}}} (\lambda \varphi(|d_i^T x'|) - \langle c_i d_i, x' \rangle) \geq -\frac{\rho}{2} |x - x'|^2,$$

which is true since $\varphi'(0+) = +\infty$ and w is C^2 smooth. Hence, h is prox-regular at \bar{x} . \square

Proposition 5.2. *Assuming the condition*

$$0 \in \partial h(\bar{x}) \text{ and } \partial^2 h(\bar{x}|0) \text{ positive definite,}$$

then it follows that h grows at least quadratically near \bar{x} . Moreover, this condition is the same as the second-order condition in Theorem 5.7.

Proof. Since

$$Z_{\bar{x}}^T \nabla w(\bar{x}) = 0 \Leftrightarrow \nabla w(\bar{x}) \in N_{\mathcal{M}}(\bar{x}) \Leftrightarrow 0 \in \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i \Leftrightarrow 0 \in \partial h(\bar{x}),$$

it follows that $0 \in \partial h(\bar{x})$ is equivalent to $Z_{\bar{x}}^T \nabla w(\bar{x}) = 0$. It is easy to see $0 \in \text{ri } \partial h(\bar{x})$ from the fact

$$\partial h(\bar{x}) = \left\{ \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} c_i d_i \mid c_i \in \mathbf{R} \right\}.$$

Since h is prox-regular at \bar{x} for $0 \in \text{ri } \partial h(\bar{x})$, and partly smooth at \bar{x} relative to \mathcal{M} , Theorem 3.6 gives us

$$\partial^2 h(\bar{x}|0)(w) = \begin{cases} \nabla^2 L(\bar{x}) + N_{\mathcal{M}}(\bar{x}) & a \in T_{\mathcal{M}}(\bar{x}) \\ \emptyset & a \notin T_{\mathcal{M}}(\bar{x}). \end{cases}$$

The function $w(x) = \theta(x) + \sum_{i \in J_{\bar{x}}} \varphi_i(|d_i^T x|)$ is a smooth representative of h restricted to \mathcal{M} . The corresponding Lagrangian function is $L(x) = w(x) + \sum_{i \in I_{\bar{x}}} \bar{\lambda}_i d_i^T x$ with $\nabla L(\bar{x}) = \nabla w(\bar{x}) + \sum_{i \in I_{\bar{x}}} \bar{\lambda}_i d_i = 0$. Hence \bar{x} being a quadratic growth minimizer is equivalent to the condition that $\nabla^2 L(\bar{x})$ is positive definite when restricted to $T_{\mathcal{M}}(\bar{x})$, i.e.,

$$a^T \nabla^2 w(\bar{x}) a > 0 \text{ for any nonzero } a \in T_{\mathcal{M}}(\bar{x}).$$

This is exactly the condition that $Z_{\bar{x}}^T \nabla^2 w(\bar{x}) Z_{\bar{x}}$ is positive definite. Therefore the theorem follows. \square

5.4 Full stability

In this section, we will discuss the stability of minimization problems from two perspectives — basic perturbations and tilt perturbations — and generalize our results in Chapter 3 to this setting. For “basic” perturbations, we use as a model a family of minimization problems over $x \in \mathbf{R}^n$ parameterized by $u \in \mathbf{R}^d$, as specified by a function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$. For “tilt” perturbations, we simply add a small linear term to the objective as we discussed in Chapter 3. Thus, we regard $\min f$ as an instance of a larger family of problems,

$$\mathcal{P}(u, v) \quad \text{minimize } f(x, u) - \langle v, x \rangle \text{ over } x \in \mathbf{R}^n,$$

with both $u \in \mathbf{R}^d$ and $v \in \mathbf{R}^n$ parameters. We consider \bar{x} , a feasible solution to $\mathcal{P}(\bar{u}, \bar{v})$, and study the functions $m_\delta : \mathbf{R}^d \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ and mappings $M_\delta : \mathbf{R}^d \times \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ (set-valued) that are defined for $\delta > 0$ by

$$\begin{aligned} m_\delta(u, v) &= \inf_{|x - \bar{x}| \leq \delta} \{ f(x, u) - \langle v, x \rangle \}, \\ M_\delta(u, v) &= \operatorname{argmin}_{|x - \bar{x}| \leq \delta} \{ f(x, u) - \langle v, x \rangle \}. \end{aligned}$$

Definition 5.1. A point \bar{x} is a *stable locally optimal solution* to $\mathcal{P}(\bar{u}, \bar{v})$ (in the basic sense, i.e., relative to the specified parameterization in u only) if there is a $\delta > 0$ such that, on some neighborhood U of \bar{u} , the mapping $u \mapsto M_\delta(u, \bar{v})$ is single-valued and Lipschitz continuous with $M_\delta(\bar{u}, \bar{v}) = \bar{x}$, and the function $u \mapsto m_\delta(u, \bar{v})$ is likewise Lipschitz continuous on U .

By contrast, the point \bar{x} is a *tilt stable locally optimal solution* if these properties hold with respect to v instead of u , i.e., for the mapping $v \mapsto M_\delta(\bar{u}, v)$ and the function $v \mapsto m_\delta(\bar{u}, v)$ on some neighborhood V of \bar{v} . It is a *fully stable locally optimal solution* if these hold with respect to (u, v) for the full mapping $(u, v) \mapsto M_\delta(u, v)$ and function $(u, v) \mapsto m_\delta(u, v)$ on some neighborhood $U \times V$ of (\bar{u}, \bar{v}) .

Full stability implies both (basic) stability and tilt stability but in general may differ from those properties. The paper [16] characterizes the full stability in terms of positive definiteness of the coderivative Hessian mapping; the characterization of tilt stability in [38] is a special case. In the setting of our parametric model, as specified by the function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$, the definitions from that paper specialize as follows.

Definition 5.2. The partial subgradient mapping $\partial_x f : \mathbf{R}^n \times \mathbf{R}^d \rightrightarrows \mathbf{R}^n$ is defined by

$$\partial_x f(x, u) = \{\text{subgradients of } f_u := f(\cdot, u) \text{ at } x\} = \partial f_u(x).$$

Definition 5.3. A lower semicontinuous function $f(x, u) : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is *prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u}* if $\bar{v} \in \partial_x f(\bar{x}, \bar{u})$, and there exist $\epsilon > 0$ and $r \geq 0$ such that

$$f(x', u) \geq f(x, u) + \langle v, x' - x \rangle - \frac{r}{2} |x' - x|^2 \text{ for all } |x' - \bar{x}| < \epsilon$$

when $x \neq x'$, $v \in \partial_x f(x, u)$, $|v - \bar{v}| < \epsilon$, $|x - \bar{x}| < \epsilon$, $|u - \bar{u}| < \epsilon$, $f(x, u) < f(\bar{x}, \bar{u}) + \epsilon$. It is

continuously prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} if, in addition, $f(x, u)$ is continuous as a function of $(x, u, v) \in \text{gph } \partial_x f$ at $(\bar{x}, \bar{u}, \bar{v})$.

Assumption 5.3. *The basic constraint qualification at a feasible solution x to $\mathcal{P}(u, v)$ is the condition*

$$\mathcal{G}(u, v) : (0, w) \in \partial^\infty f(x, u) \Rightarrow w = 0.$$

Definition 5.4. *For any mapping $S : \mathbf{R}^m \rightrightarrows \mathbf{R}^p$, we denote by $\text{gph } S$ the set of all pairs $(z, w) \in \mathbf{R}^m \times \mathbf{R}^p$ such that $w \in S(z)$. For any such pair (z, w) , the coderivative of S at z for w is the mapping $D^*S(z|w) : \mathbf{R}^p \rightrightarrows \mathbf{R}^m$ defined by*

$$D^*S(z|w)(w') = \{z' \mid (z', -w') \in N_{\text{gph } S}(z, w)\}.$$

When S is single-valued and C^1 around z with Jacobian matrix $\nabla S(z)$, the coderivative for $w = S(z)$ reduces to the adjoint linear mapping $w' \mapsto \nabla S(z)^* w'$.

The full stability characterization will center on the property of the coderivative mapping $D^*(\partial_x f)(x, u|v)$. For clarity, it is worth contrasting that mapping with the following mapping:

$$\partial^2 f_x(x, u|v) := D^*(\partial f_u)(x|v) = \partial^2 f_u(x|v)$$

(where we recall $f_u = f(\cdot, u)$). To illustrate, if f is C^2 smooth and $v = \nabla_x f(x, u)$, then

$$\partial^2_x f(x, u|v)w = \nabla_{xx}^2 f(x, u)w$$

while

$$D^*(\partial_x f)(x, u|v)w = (\nabla_{xx}^2 f(x, u)w, \nabla_{ux}^2 f(x, u)w).$$

The following theorem gives a characterization of full stability.

Theorem 5.8. *Let \bar{x} be a feasible solution to $\mathcal{P}(\bar{u}, \bar{v})$ at which the first-order condition $\bar{v} \in \partial_x f(\bar{x}, \bar{u})$ is satisfied along with the constraint qualification $\mathcal{G}(\bar{x}, \bar{u})$. Suppose $f(x, u)$*

is continuously prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} . Then for \bar{x} to be a locally optimal solution to $\mathcal{P}(\bar{u}, \bar{v})$ that is fully stable, it is necessary and sufficient that the following second-order conditions be fulfilled:

$$(a) \quad (x', u') \in D^*(\partial_x f)(\bar{x}, \bar{u}|\bar{v})(v'), v' \neq 0 \Rightarrow \langle v', x' \rangle > 0,$$

$$(b) \quad (0, u') \in D^*(\partial_x f)(\bar{x}, \bar{u}|\bar{v})(0) \Rightarrow u' = 0.$$

Proof. See [16, Thm. 2.3]. □

Our goal in this section is to explore the above theorem by only considering partly smooth and prox-regular functions.

Assumption 5.4. For Euclidean spaces \mathbf{R}^n and \mathbf{R}^d , the set $Q \subset \mathbf{R}^n \times \mathbf{R}^d$ is a manifold containing the point (\bar{x}, \bar{u}) and satisfying the condition

$$(0, w) \in N_Q(\bar{x}, \bar{u}) \Rightarrow w = 0.$$

Note that Assumption 5.4 implies that the same assumption also holds at all points in Q near (\bar{x}, \bar{u}) .

Proposition 5.3. Let $Q_u = \{x \in X : (x, u) \in Q\}$. If Assumption 5.4 holds, then there is an open neighborhood $X \times U$ of (\bar{x}, \bar{u}) such that for all vectors $u \in U$ the set $Q_u \cap X$ is a manifold.

Proof. See [17, Prop. 5.2]. □

We can be more concrete, as follows.

Lemma 5.3. Suppose that $Q \subset \mathbf{R}^n \times \mathbf{R}^d$ is a C^2 manifold around (\bar{x}, \bar{u}) (with codimension r) and Assumption 5.4 holds at (\bar{x}, \bar{u}) , so there exists an open set $X \times U \subset \mathbf{R}^n \times \mathbf{R}^d$ such that

$$Q \cap (X \times U) = \{(x, u) \in X \times U : \Phi_i(x, u) = 0, i = 1, \dots, r\},$$

where $\nabla \Phi_i$ are linearly independent for all $(x, u) \in Q \cap (X \times U)$. Then for any $(x, u) \in Q \cap (X \times U)$ we have

$$T_Q(x, u) = \{\nabla \Phi_i(x, u)\}^\perp \text{ and } N_Q(x, u) = \left\{ \sum_i \lambda_i \nabla \Phi_i(x, u) \mid \lambda \in \mathbf{R}^r \right\}.$$

Furthermore, for any $u \in U$, the set

$$Q_u \cap X = \{x \in X : \Phi_i(x, u) = 0, i = 1, \dots, r\}$$

is a manifold, with $\nabla_x \Phi_i(x, u)$ are linearly independent, and

$$T_{Q_u}(x) = \{\nabla_x \Phi_i(x, u)\}^\perp \text{ and } N_{Q_u}(x) = \left\{ \sum_i \lambda_i \nabla_x \Phi_i(x, u) : \lambda \in \mathbf{R}^r \right\}$$

for all $x \in Q_u \cap X$.

Proof. Without loss of generality, we can assume that for all points (x, u) in $X \times U$ Assumption 5.4 is satisfied and $\nabla \Phi(x, u)$ are independent. Now it is sufficient to prove that $\nabla_x \Phi_i(x, u)$ are linearly independent for all $x \in Q_u \cap X$. If $\sum \lambda_i \nabla_x \Phi_i(x, u) = 0$, then $(0, \sum \lambda_i \nabla_u \Phi_i(x, u)) = (\sum \lambda_i \nabla_x \Phi_i(x, u), \sum \lambda_i \nabla_u \Phi_i(x, u)) \in N_Q(x, u)$. Since (x, u) satisfies Assumption 5.4 and $\nabla \Phi_i(x, u)$ are linearly independent, then $\lambda = 0$. Therefore, $\nabla_x \Phi_i(x, u)$ are linearly independent, and the result follows. \square

Proposition 5.4.

$$N_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v}) = \left\{ (x', u', v') \mid v' \in T_{Q_{\bar{u}}}(\bar{x}), \right. \\ \left. \left(x' + \sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}) v', u' + \sum_i \bar{\lambda}_i \nabla_{ux}^2 \Phi_i(\bar{x}, \bar{u}) v' \right) \in N_Q(\bar{x}, \bar{u}) \right\}.$$

Furthermore,

$$D^*(\partial_x \delta_Q(\bar{x}, \bar{u}|\bar{v}))(v') = \begin{cases} \left(\sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}) v', \sum_i \bar{\lambda}_i \nabla_{ux}^2 \Phi_i(\bar{x}, \bar{u}) v' \right) + N_Q(\bar{x}, \bar{u}) & v' \in T_{Q_{\bar{u}}}(\bar{x}) \\ \emptyset & v' \notin T_{Q_{\bar{u}}}(\bar{x}) \end{cases}$$

where $\bar{v} = \sum_i \bar{\lambda}_i \nabla_x \Phi_i(\bar{x}, \bar{u})$.

Proof. Since Q is a C^2 manifold around (\bar{x}, \bar{u}) of codimension r , there exists a neighborhood $V \subset \mathbf{R}^{n+d-r}$ and an injective C^2 mapping $G : V \rightarrow \mathbf{R}^n \times \mathbf{R}^d$ with $\nabla G(\bar{x}, \bar{u})$ injective and $G(V) = Q$ locally around (\bar{x}, \bar{u}) , by Proposition 3.1. Define a map $F : V \times \mathbf{R}^r \rightarrow \mathbf{R}^{n+d} \times \mathbf{R}^n$ by

$$F(a, b, \lambda) = \left(G(a, b), \sum_i \lambda_i \nabla_x \Phi_i(G(a, b)) \right)$$

with $F(\bar{a}, \bar{b}, \bar{\lambda}) = (\bar{x}, \bar{u}, \bar{v})$ and $F(V \times \mathbf{R}^r) = \text{gph } N_{Q_{\bar{u}}}$ locally around $(\bar{x}, \bar{u}, \bar{v})$.

We first want to compute $T_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v})$. We know that

$$\begin{aligned} T_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v}) &= R(\nabla F(\bar{a}, \bar{b}, \bar{\lambda})) \\ &= \left\{ (x', u', v') \mid (x', u') \in T_Q(\bar{x}, \bar{u}), \right. \\ &\quad \left. v' - \left(\sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}), \sum_i \bar{\lambda}_i \nabla_{xu}^2 \Phi_i(\bar{x}, \bar{u}) \right) (x', u')^T \in N_{Q_{\bar{u}}}(\bar{x}) \right\}. \end{aligned}$$

We can calculate the normal cone from the fact that for any linear map A and a linear subspace S then we have

$$\{ x \mid Ax \in S \}^\perp = A^* S^\perp.$$

In this case,

$$A = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_d & 0 \\ -\sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}) & -\sum_i \bar{\lambda}_i \nabla_{xu}^2 \Phi_i(\bar{x}, \bar{u}) & I_n \end{pmatrix}$$

and

$$S = T_Q(\bar{x}, \bar{u}) \times N_{Q_{\bar{u}}}(\bar{x}).$$

Hence we deduce

$$\begin{aligned}
N_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v}) &= T_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v})^\perp \\
&= \left\{ (x', u', v') \mid v' \in T_{Q_{\bar{u}}}(\bar{x}), \right. \\
&\quad \left. \left(x' + \sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}) v', u' + \sum_i \bar{\lambda}_i \nabla_{ux}^2 \Phi_i(\bar{x}, \bar{u}) v' \right) \in N_Q(\bar{x}, \bar{u}) \right\}.
\end{aligned}$$

By definition we know

$$(x', u') \in D^*(\partial_x \delta_Q)(\bar{x}, \bar{u} | \bar{v})(v') \Leftrightarrow (x', u', -v') \in N_{\text{gph } N_{Q_{\bar{u}}}}(\bar{x}, \bar{u}, \bar{v}),$$

so we deduce

$$D^*(\partial_x \delta_Q)(\bar{x}, \bar{u} | \bar{v})(v') = \begin{cases} \left(\sum_i \bar{\lambda}_i \nabla_{xx}^2 \Phi_i(\bar{x}, \bar{u}) v', \sum_i \bar{\lambda}_i \nabla_{ux}^2 \Phi_i(\bar{x}, \bar{u}) v' \right) + N_Q(\bar{x}, \bar{u}) & v' \in T_{Q_{\bar{u}}}(\bar{x}) \\ \emptyset & v' \notin T_{Q_{\bar{u}}}(\bar{x}) \end{cases}$$

where $\bar{v} = \sum_i \bar{\lambda}_i \nabla_x \Phi_i(\bar{x}, \bar{u})$. \square

We next extend by adding a smooth function.

Proposition 5.5. *Let $h : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \mathbf{R}$ be a C^2 function. Define the function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ by $f(x, u) = h(x, u) + \delta_Q(x, u)$ with Q satisfying Assumption 5.4 at (\bar{x}, \bar{u}) .*

Then

$$D^*(\partial_x f)(\bar{x}, \bar{u} | \bar{v})(v') = \begin{cases} \left(\nabla_{xx}^2 h(\bar{x}, \bar{u}) v' + \sum_j \bar{\lambda}_j \nabla_{xx}^2 \Phi_j(\bar{x}, \bar{u}) v', \right. \\ \quad \left. \nabla_{ux}^2 h(\bar{x}, \bar{u}) v' + \sum_j \bar{\lambda}_j \nabla_{ux}^2 \Phi_j(\bar{x}, \bar{u}) v' \right) + N_Q(\bar{x}, \bar{u}) & v' \in T_{Q_{\bar{u}}}(\bar{x}) \\ \emptyset & v' \notin T_{Q_{\bar{u}}}(\bar{x}) \end{cases}$$

where $\nabla_x h(\bar{x}, \bar{u}) + \sum_j \bar{\lambda}_j \nabla_x \Phi_j(\bar{x}, \bar{u}) = \bar{v}$.

Proof. Since Assumption 5.4 holds at (\bar{x}, \bar{u}) , there exists a neighborhood of (\bar{x}, \bar{u}) such that $\partial_x f(x, u) = \nabla_x h(x, u) + \partial_x \delta_Q(x, u)$ for any point (x, u) in this neighborhood. According to [43, 10.43], we have

$$D^*(\partial_x f)(\bar{x}, \bar{u} | \bar{v})(v') = D^*(\partial_x \delta_Q)(\bar{x}, \bar{u} | \bar{v} - \nabla_x h(\bar{x}, \bar{u}))(v') + (\nabla_{xx}^2 h(\bar{x}, \bar{u}), \nabla_{ux}^2 h(\bar{x}, \bar{u})) v'.$$

The result now follows. \square

Next, we will generalize the above result to prox-regular and partly smooth functions. We use the following tool.

Proposition 5.6. *For a proper, lsc function $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \overline{\mathbf{R}}$ and a point $(\bar{x}, \bar{u}) \in \text{dom } f$, let $\partial_x f(\bar{x}, \bar{u})$ denote the subgradients of $f(\cdot, \bar{u})$ at \bar{x} , and similarly $\hat{\partial}_x f(\bar{x}, \bar{u})$. One always has*

$$\hat{\partial}_x f(\bar{x}, \bar{u}) \supset \{v \mid \exists y \text{ with } (v, y) \in \hat{\partial} f(\bar{x}, \bar{u})\}.$$

On the other hand, under the condition that $(0, y) \in \partial^\infty f(\bar{x}, \bar{u})$ implies $y = 0$, one also has

$$\partial_x f(\bar{x}, \bar{u}) \subset \{v \mid \exists y \text{ with } (v, y) \in \partial f(\bar{x}, \bar{u})\}.$$

If in addition f is regular at (\bar{x}, \bar{u}) , then $f(\cdot, \bar{u})$ is regular at \bar{x} and

$$\partial_x f(\bar{x}, \bar{u}) = \{v \mid \exists y \text{ with } (v, y) \in \partial f(\bar{x}, \bar{u})\}.$$

Proof. See [43, Cor. 10.11]. □

Proposition 5.7. *Let Q be a C^2 -smooth manifold with $(\bar{x}, \bar{u}) \in Q$ satisfying Assumption 5.4. Suppose function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \overline{\mathbf{R}}$ is partly smooth relative to the manifold Q . Then there is an open neighborhood $X \times U$ of (\bar{x}, \bar{u}) such that the function f_u is partly smooth relative to $Q_u \cap X$ for all $u \in U$. Moreover, $\partial_x f(x, u) = P_{\mathbf{R}^n} \partial f(x, u)$ for all $(x, u) \in (X \times U) \cap Q$.*

Proof. The first part of this statement follows from [17, Prop. 5.3]. Since f is partly smooth at (x, u) relative to Q for all $(x, u) \in Q$, then $\partial^\infty f(x, u) \subset N_Q(x, u)$. Assumption 5.4 implies that the only choice of $(0, y) \in \partial^\infty f(x, u) \subset N_Q(x, u)$ with $(x, u) \in (X \times U) \cap Q$ is $y = 0$. Hence $\partial_x f(x, u) = P_{\mathbf{R}^n} \partial f(x, u)$ for all $(x, u) \in (X \times U) \cap Q$, using Proposition 5.6. □

Proposition 5.8. *Let Q be a C^2 -smooth manifold with (\bar{x}, \bar{u}) satisfying Assumption 5.4. Suppose the function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is partly smooth relative to the manifold Q with $\bar{v} \in \text{ri } \partial_x f(\bar{x}, \bar{u})$. Let h be any C^2 -smooth representative of f around (\bar{x}, \bar{u}) . Then there exists $\epsilon_1 > 0$ such that for all $(x, u) \in Q$ and $v \in \nabla_x h(x, u) + N_{Q_u}(x)$ with $|(x, u, v) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_1$, we have $v \in \partial_x f(x, u)$.*

Proof. By Proposition 5.7, we know that there exists a neighborhood $X \times U$ of (\bar{x}, \bar{u}) such that $Q_u \cap X$ is a manifold and f_u is partly smooth relative to $Q_u \cap X$ for all $u \in U$ and $\partial_x f(x, u) = P_{\mathbf{R}^n} \partial f(x, u)$ for all $(x, u) \in (X \times U) \cap Q$. Then we have

$$\nabla_x h(x, u) + N_{Q_u}(x) = \text{aff } \partial_x f(x, u) \text{ for } (x, u) \in (X \times U) \cap Q.$$

If the required ϵ_1 doesn't exist, then there exists a sequence $(x_n, u_n, v_n) \rightarrow (\bar{x}, \bar{u}, \bar{v})$ such that $(x_n, u_n) \in Q$, $v_n \in \nabla_x h(x_n, u_n) + N_{Q_{u_n}}(x_n)$ and $v_n \notin \partial_x f(x_n, u_n)$. Since $f(\cdot, u_n)$ is regular at x_n when n is large, it follows that $\partial_x f(x_n, u_n)$ is closed and convex. According to the Separation Theorem, for all large n there exists a unit vector $z_n \in \text{par } \partial_x f(x_n, u_n) = N_{Q_{u_n}}(x_n)$ such that

$$\langle z_n, v \rangle \geq \langle z_n, v_n \rangle$$

for all $v \in \partial_x f(x_n, u_n)$. Passing to a subsequence if necessary, we can assume z_n approaches a nonzero vector z . Since $\partial f(x_n, v_n) \rightarrow \partial f(\bar{x}, \bar{u})$, it follows that $\partial_x f(x_n, v_n) \rightarrow \partial_x f(\bar{x}, \bar{u})$. Therefore $N_{Q_{u_n}}(x_n) \rightarrow N_{Q_{\bar{u}}}(\bar{x})$. As a result, we have

$$z \in \partial_x f(\bar{x}, \bar{u}) \text{ and } \langle z, v \rangle \geq \langle z, \bar{v} \rangle$$

for any $v \in \partial_x f(\bar{x}, \bar{u})$, which shows that \bar{v} is separated from the convex set $\partial_x f(\bar{x}, \bar{u})$ in its affine span. But this contradicts the fact $\bar{v} \in \text{ri } \partial_x f(\bar{x}, \bar{u})$. Therefore, the result follows. \square

We next turn to the question of prox regularity.

Proposition 5.9. *Let Q be a C^2 -smooth manifold with (\bar{x}, \bar{u}) satisfying Assumption 5.4. Suppose the function $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is partly smooth relative to the manifold Q with $\bar{v} \in \text{ri } \partial_x f(\bar{x}, \bar{u})$. Moreover, suppose the function f is prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} with respect to r and ϵ_2 .*

1. *There exists $\epsilon_3 > 0$ such that for any $(x_0, u_0) \in Q$ and $v_0 \in \partial_x f(x_0, u_0)$ satisfying $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_3$, the function f is prox-regular at x_0 for v_0 with compatible parameterization by u at u_0 with respect to ϵ_3 and r .*
2. *There exists $\epsilon_4 > 0$ such that for any $(x_0, u_0) \in Q$ and $v_0 \in \partial_x f(x_0, u_0)$ satisfying $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_4$, we have*

$$P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)(x) = P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle + \delta_{Q_{u_0}})(x)$$

on the set $X(x_0, \epsilon_4) := \{x : |x - x_0| < \epsilon_4\}$ for any $\lambda \in (0, 1/r)$.

Proof. (1). Since f is prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} with respect to r and ϵ_2 , we know

$$f(x', u) > f(x, u) + \langle v, x' - x \rangle - \frac{r}{2}|x - x'|^2 \text{ whenever } |x' - \bar{x}| < \epsilon_2$$

providing $x \neq x'$, $|x - \bar{x}| < \epsilon_2$, $|u - \bar{u}| < \epsilon_2$, $f(x, u) < f(\bar{x}, \bar{u}) + \epsilon_2$ and $|v - \bar{v}| < \epsilon_2$ with $v \in \partial_x f(x, u)$. Since $f(x, u)$ is continuous relative to Q , there exists $\hat{\epsilon} > 0$ such that $|(x_0, u_0) - (\bar{x}, \bar{u})| < \hat{\epsilon}$ with $(x_0, u_0) \in Q$ implies $f(x_0, u_0) < f(\bar{x}, \bar{u}) + \frac{\epsilon_2}{2}$. Let $\epsilon_3 = \min\{\frac{\epsilon_2}{2}, \hat{\epsilon}\}$. Then we claim that for any $(x_0, u_0) \in Q$ and $v_0 \in \partial_x f(x_0, u_0)$ satisfying $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_3$, the function f is prox-regular in x at x_0 for v_0 with compatible parameterization by u at u_0 with respect to ϵ_3 and r . Providing $|x' - x_0| < \epsilon_3$, $|x - x_0| < \epsilon_3$, $|u - u_0| < \epsilon_3$, $|v - v_0| < \epsilon_3$ with $v \in \partial_x f(x, u)$ and $f(x, u) < f(x_0, u_0) + \epsilon_3$, we have $|x' - \bar{x}| \leq |x' - x_0| + |x_0 - \bar{x}| < \epsilon_3 + \epsilon_3 \leq \epsilon_2$, similarly $|x - \bar{x}| < \epsilon_2$, $|v - \bar{v}| < \epsilon_2$, $|u - \bar{u}| < \epsilon_2$ and $f(x, u) < f(\bar{x}, \bar{u}) + \epsilon_2$. Therefore we have

$$f(x', u) > f(x, u) + \langle v, x' - x \rangle - \frac{r}{2}|x - x'|^2 \text{ for all } |x_0 - x'| < \epsilon_3$$

when $x' \neq x$, $|x - x_0| < \epsilon_3$, $|v - v_0| < \epsilon_3$ with $v \in \partial_x f(x, u)$ and $f(x, u) < f(x_0, u_0) + \epsilon_3$. Therefore, the claim follows.

(2). For any $(x_0, u_0) \in Q$ and $v_0 \in \partial_x f(x_0, u_0)$ satisfying $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_3$, without loss of generality, we can assume that both $f_{u_0}(x) - \langle v_0, x \rangle$ and $f_{u_0}(x) - \langle v_0, x \rangle + \delta_{Q_{u_0}}(x)$ are prox-regular at x_0 for 0 with respect to r and ϵ_3 .

Let $T_{(u_0, v_0)}$ be the $f_{u_0} - \langle v_0, \cdot \rangle$ -attentive ϵ_3 -localization of $\partial_x f(x_0, u_0) - v_0$ around $(x_0, 0)$. For any $\lambda \in (0, \frac{1}{r})$, there exists $\gamma > 0$ such that both $P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)$ and $P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle + \delta_{Q_{u_0}})$ are single valued and Lipschitz continuous with constant $\frac{1}{1-\lambda r}$ on $X_\gamma(x_0) := \{x \mid |x - x_0| < \gamma\}$ by [37, Thm. 4.4], whose proof also implies that the γ is only determined by parameters r and ϵ_3 . Moreover,

$$P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)(x) = (I + \lambda T_{(u_0, v_0)})^{-1}(x) \text{ on } X_\gamma(x_0).$$

There exists $\delta > 0$ such that, providing $|(x, u, v) - (\bar{x}, \bar{u}, \bar{v})| < \delta$ with $(x, u) \in Q$, we have $|f(x, u) - \langle v, x \rangle - f(\bar{x}, \bar{u}) + \langle \bar{v}, \bar{x} \rangle| < \frac{\epsilon_3}{2}$. Let $\epsilon_4 < \min\{C\delta, \gamma, C\epsilon_1, C\epsilon_3\}$ for a constant $C \geq 1$ to be specified later. Then providing $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_4$ and $|x - x_0| < \epsilon_4$, we have

$$w_{(u_0, v_0)} = P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle + \delta_{Q_{u_0}})(x) = \operatorname{argmin}_w \{h(w, u_0) - \langle v_0, w \rangle + \delta_{Q_{u_0}}(w) + \frac{1}{2\lambda}|w - x|^2\} \in Q_{u_0}$$

where h is a C^2 smooth representative of f around (\bar{x}, \bar{u}) . Also we have

$$P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)(x_0) = x_0.$$

Since $w_{(u_0, v_0)}$ minimizes $h(w, u_0) - \langle v_0, w \rangle + \delta_{Q_{u_0}}(w) + \frac{1}{2\lambda}|w - x|^2$, we know

$$0 \in \nabla_x h(w_{(u_0, v_0)}, u_0) - v_0 + N_{Q_{u_0}}(w_{(u_0, v_0)}) + \frac{1}{\lambda}(w_{(u_0, v_0)} - x)$$

or equivalently

$$v_0 + \frac{1}{\lambda}(x - w_{(u_0, v_0)}) \in \nabla h_x(w_{(u_0, v_0)}, u_0) + N_{Q_{u_0}}(w_{(u_0, v_0)}).$$

Hence

$$\begin{aligned}
& |(w_{(u_0, v_0)}, u_0, v_0 + \frac{1}{\lambda}(x - w_{(u_0, v_0)})) - (\bar{x}, \bar{u}, \bar{v})| \\
& \leq |(w_{(u_0, v_0)}, u_0, v_0 + \frac{1}{\lambda}(x - w_{(u_0, v_0)})) - (x_0, u_0, v_0)| + |(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| \\
& \leq |w_{(u_0, v_0)} - x_0| + \frac{1}{\lambda}|x - w_{(u_0, v_0)}| + \epsilon_4 \\
& \leq \frac{1}{1 - \lambda r}|x - x_0| + \frac{1}{\lambda}|x - x_0| + \frac{1}{\lambda}|x_0 - w_{(u_0, v_0)}| + \epsilon_4 \\
& \leq (\frac{\lambda + 1}{\lambda(1 - \lambda r)} + \frac{1}{\lambda})|x - x_0| + \epsilon_4 \\
& \leq (\frac{\lambda + 1}{\lambda(1 - \lambda r)} + \frac{1}{\lambda} + 1)\epsilon_4 \text{ let } C := \frac{\lambda + 1}{\lambda(1 - \lambda r)} + \frac{1}{\lambda} + 1 \\
& < \epsilon_1.
\end{aligned}$$

According to Proposition 5.8, we have

$$v_0 + \frac{1}{\lambda}(x - w_{(u_0, v_0)}) \in \partial f_{u_0}(w_{(u_0, v_0)}).$$

Moreover,

$$\begin{aligned}
\frac{1}{\lambda}|x - w_{(u_0, v_0)}| & \leq \frac{1}{\lambda}(|x - x_0| + |x_0 - w_{(u_0, v_0)}|) \\
& \leq \frac{1}{\lambda}(1 + \frac{1}{1 - \lambda r})|x - x_0| \\
& \leq C\epsilon_4 \\
& < \epsilon_3.
\end{aligned}$$

Since

$$\begin{aligned}
& |(w_{(u_0, v_0)}, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| \\
& \leq |(w_{(u_0, v_0)}, u_0, v_0) - (x_0, u_0, v_0)| + |(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| \\
& \leq (1 + \frac{1}{1 - \lambda r})|x - x_0| + \epsilon_4 \\
& \leq (2 + \frac{1}{1 - \lambda r})\epsilon_4 \leq C\delta \leq \delta,
\end{aligned}$$

we deduce

$$\begin{aligned}
& |f(w_{(u_0, v_0)}, u_0) - \langle v_0, w_{(u_0, v_0)} \rangle - f(x_0, u_0) + \langle v_0, x_0 \rangle| \\
& \leq |f(w_{(u_0, v_0)}, u_0) - \langle v_0, w_{(u_0, v_0)} \rangle - f(\bar{x}, \bar{u}) + \langle \bar{v}, \bar{x} \rangle| \\
& \quad + |f(\bar{x}, \bar{u}) - \langle \bar{v}, \bar{x} \rangle - f(x_0, u_0) + \langle v_0, x_0 \rangle| \\
& < \frac{\epsilon_3}{2} + \frac{\epsilon_3}{2} = \epsilon_3.
\end{aligned}$$

Therefore, we have

$$\frac{1}{\lambda}(x - w_{(u_0, v_0)}) \in T_{(u_0, v_0)}(w_{(u_0, v_0)}).$$

Thus $x \in (I + \lambda T_{(u_0, v_0)})(w_{(u_0, v_0)})$, so $w_{u_0} \in (I + \lambda T_{(u_0, v_0)})^{-1}(x) = P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)(x)$.

Therefore

$$P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle)(x) = P_\lambda(f_{u_0} - \langle v_0, \cdot \rangle + \delta_{Q_{u_0}})(x) \text{ on } X(x_0, \epsilon_4),$$

providing $|(x_0, u_0, v_0) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_4$ with $(x_0, u_0) \in Q$ and $v_0 \in \partial_x f(x_0, u_0)$. \square

We now arrive at a parametric identifiability result.

Proposition 5.10. *Let Q be a C^2 -manifold with (\bar{x}, \bar{u}) satisfying Assumption 5.4. Suppose that $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth relative to Q with $\bar{v} \in \text{ri } \partial_x f(\bar{x}, \bar{u})$. Suppose f is prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} with respect to r and ϵ_2 , and furthermore $f(x, u)$ is continuous as a function of $(x, u, v) \in \text{gph } \partial_x f$ at $(\bar{x}, \bar{u}, \bar{v})$. Then for any $(x_k, u_k, v_k) \in \text{gph } \partial_x f(x_k, u_k) \rightarrow (\bar{x}, \bar{u}, \bar{v})$ with $v_k \rightarrow \bar{v}$, then we have $(x_k, u_k) \in Q$ for all large k .*

Proof. Since $(x_k, u_k) \rightarrow (\bar{x}, \bar{u}) \in Q$, then there exists $\bar{x}_k \in Q_{u_k}$ such that $(\bar{x}_k, u_k) \in Q \rightarrow (\bar{x}, \bar{v})$ for large k . Since $\partial f(x, u)$ is continuous at (\bar{x}, \bar{u}) relative to Q , then there exists $\bar{v}_k \in \partial_x f(\bar{x}_k, u_k)$ such that $(\bar{x}_k, u_k, \bar{v}_k) \rightarrow (\bar{x}, \bar{u}, \bar{v})$. Also, $f(x_k, u_k) \rightarrow f(\bar{x}, \bar{u})$ and $f(\bar{x}_k, u_k) \rightarrow f(\bar{x}, \bar{u})$ implies that $f(x_k, u_k) - f(\bar{x}_k, u_k) \rightarrow 0$. For large k , we have

$$|(\bar{x}_k, u_k, \bar{v}_k) - (\bar{x}, \bar{u}, \bar{v})| < \epsilon_4, |x_k - \bar{x}_k + \lambda(v_k - \bar{v}_k)| < \epsilon_4.$$

Hence

$$\begin{aligned}
& \frac{1}{\lambda}((x_k + \lambda v_k) - x_k) \in \partial f_{u_k}(x_k) \\
\Rightarrow & \frac{1}{\lambda}((x_k + \lambda v_k) - x_k) - \bar{v}_k \in \partial f_{u_k}(x_k) - \bar{v}_k \\
\Rightarrow & \frac{1}{\lambda}((x_k + \lambda v_k) - x_k) - \bar{v}_k \in T_{(u_k, \bar{v}_k)}(x_k) \\
\Rightarrow & x_k + \lambda(v_k - \bar{v}_k) \in (I + \lambda T_{(u_k, \bar{v}_k)})(x_k),
\end{aligned}$$

where $T_{(u_k, \bar{v}_k)}$ denotes the $f_{u_k} - \langle \bar{v}_k, \cdot \rangle$ -attentive ϵ_3 localization of the partial subdifferential around (\bar{x}_k, \bar{v}_k) . According to Proposition 5.9, we have

$$\begin{aligned}
x_k & \in (I + \lambda T_{(u_k, \bar{v}_k)})^{-1}(x_k + \lambda(v_k - \bar{v}_k)) \\
& = P_\lambda(f_{u_k} - \langle \bar{v}_k, \cdot \rangle)(x_k + \lambda(v_k - \bar{v}_k)) \\
& = P_\lambda(f_{u_k} + \delta_{Q_{u_k}} - \langle \bar{v}_k, \cdot \rangle)(x_k + \lambda(v_k - \bar{v}_k)).
\end{aligned}$$

Hence $x_k \in Q_{u_k}$ for all large k . □

Summarizing, the characterization for full stability gives us the following theorem.

Theorem 5.9. *Let Q be a C^2 -manifold with (\bar{x}, \bar{u}) satisfying Assumption 5.4. Suppose that $f : \mathbf{R}^n \times \mathbf{R}^d \rightarrow \bar{\mathbf{R}}$ is C^2 -partly smooth relative to Q with $\bar{v} \in \text{ri } \partial_x f(\bar{x}, \bar{u})$, and f is prox-regular in x at \bar{x} for \bar{v} with compatible parameterization by u at \bar{u} , and continuous as a function of $(x, u, v) \in \text{gph } \partial_x f$ at $(\bar{x}, \bar{u}, \bar{v})$. Let h be any C^2 -smooth representative of f around (\bar{x}, \bar{u}) . Then for \bar{x} to be a locally optimal solution to $\min_x f(x, \bar{u}) - \langle \bar{v}, x \rangle$ that is fully stable, it is necessary and sufficient that the following conditions are fulfilled:*

- (a) *There exists $\bar{\lambda} \in \mathbf{R}^m$ with $\nabla_x h(\bar{x}, \bar{u}) + \sum_j \bar{\lambda}_j \nabla_x \Phi_j(\bar{x}, \bar{u}) = \bar{v}$;*
- (b) *For any $0 \neq v' \in T_{Q_u}(\bar{x})$ one has*

$$\langle v', (\nabla_{xx}^2 h(\bar{x}, \bar{u}) + \sum_j \bar{\lambda}_j \nabla_{xx}^2 \Phi_j(\bar{x}, \bar{u}))v' \rangle > 0.$$

Proof. By the above proposition, we know $\text{gph } \partial_x f = \text{gph } \partial_x(h + \delta_Q)$ around $(\bar{x}, \bar{u}, \bar{v})$. According to Proposition 5.5, we know that

$$D^*(\partial_x f(\bar{x}, \bar{u}|\bar{v}))(v') = \begin{cases} \left(\nabla_{xx}^2 h(\bar{x}, \bar{u})v' + \sum_j \bar{\lambda}_j \nabla_{xx}^2 \Phi_j(\bar{x}, \bar{u})v' \right. \\ \left. \nabla_{ux}^2 h(\bar{x}, \bar{u})v' + \sum_j \bar{\lambda}_j \nabla_{ux}^2 \Phi_j(\bar{x}, \bar{u})v' \right) + N_Q(\bar{x}, \bar{u}) & v' \in T_{Q_u}(\bar{x}) \\ \emptyset & v' \notin T_{Q_u}(\bar{x}) \end{cases}$$

Then the theorem follows by Theorem 5.8. \square

Note that these conditions are equivalent to \bar{x} being a tilt stable local minimum for $f(x, u) - \langle \bar{v}, x \rangle$. In this partly smooth setting, tilt stability implies full stability under reasonable conditions, although in general it may differ.

5.5 Future research

From these starting points, there are many interesting directions to pursue.

An extensive theory has been established for partly smooth functions, but practical algorithms for partly smooth functions have been little studied. The manuscript [20] introduced a proximal method for composite minimization and proposed an algorithmic framework via a proximal linearized subproblem. Further investigation of this method and how it works in reality would be interesting. The paper [25] designed a “ \mathcal{UV} ”-algorithm for convex minimization. Since partly smooth functions also have a \mathcal{UV} decomposition, it would be of interest to see if that idea can be extended to partly smooth functions.

Questions abound on the limiting behavior of nonsmooth BFGS methods. Any progress on the convergence conjecture in [19] would be fascinating. Plausible directions might involve proving convergence results for certain classes of

nonsmooth functions (such as pointwise maxima of two smooth convex functions), analyzing the reasons why line-search BFGS somehow “identifies” the active manifold for partly smooth functions, or directly seeking counterexamples.

APPENDIX A
CHAPTER 4 OF APPENDIX

A.1 Proof of Proposition 4.2

Proof of Proposition 4.2. Let $\rho = \frac{a}{(a+1)^2}$. We have

$$p_0 = -H_0 \nabla f_0 = \begin{pmatrix} -\frac{a}{a+1} \\ -\frac{2}{(a+1)^2} \end{pmatrix} \quad y_0 = \nabla f_1 - \nabla f_0 = \begin{pmatrix} -\frac{2a}{a+1} \\ -(a+1) \end{pmatrix}$$

$$\begin{aligned} V_0 &= I - (p_0^T y_0) p_0 y_0^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{(a+1)^2}{2(a^2+a+1)} \begin{pmatrix} \frac{2a^2}{(a+1)^2} & a \\ \frac{4a}{(a+1)^3} & \frac{2}{a+1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{a(a+1)^2}{2(a^2+a+1)} \\ -\frac{2a}{(a+1)(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \end{aligned}$$

$$\alpha_0 = \operatorname{argmin} \left\{ \left(1 - \frac{a}{a+1} \alpha\right)^2 - a \left(\frac{2}{a^2+3a+1} - \frac{2}{(a+1)^2} \alpha \right) \right\} = 1.$$

$$\begin{aligned} H_1 &= V_0 H_0 V_0^T + \alpha_0 (p_0^T y_0)^{-1} p_0 p_0^T \\ &= \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{a(a+1)^2}{2(a^2+a+1)} \\ -\frac{2a}{(a+1)(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \begin{pmatrix} \frac{a}{2(a+1)} & 0 \\ 0 & \frac{2}{(a+1)^2} \end{pmatrix} \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{2a}{(a+1)(a^2+a+1)} \\ -\frac{a(a+1)^2}{2(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \\ &\quad + \frac{(a+1)^2}{2a^2+2a+2} \begin{pmatrix} \frac{a^2}{(a+1)^2} & \frac{2a}{(a+1)^3} \\ \frac{2a}{(a+1)^3} & \frac{4}{(a+1)^4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a(a+1)}{2(a^2+a+1)} & -\frac{a^2}{(a+1)(a^2+a+1)} \\ -\frac{a^2}{(a+1)(a^2+a+1)} & \frac{2a^3}{(a+1)^3(a^2+a+1)} \end{pmatrix} + \begin{pmatrix} \frac{a^2}{2(a^2+a+1)} & \frac{a}{(a+1)(a^2+a+1)} \\ \frac{a}{(a+1)(a^2+a+1)} & \frac{2}{(a+1)^2(a^2+a+1)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2a^2+a}{2(a^2+a+1)} & \frac{a-a^2}{(a+1)(a^2+a+1)} \\ \frac{a-a^2}{(a+1)(a^2+a+1)} & \frac{2(a^3+a+1)}{(a+1)^3(a^2+a+1)} \end{pmatrix} \end{aligned}$$

$$p_1 = -H_1 \nabla f_1 = - \begin{pmatrix} \frac{2a^2+a}{2(a^2+a+1)} & \frac{a-a^2}{(a+1)(a^2+a+1)} \\ \frac{a-a^2}{(a+1)(a^2+a+1)} & \frac{2(a^3+a+1)}{(a+1)^3(a^2+a+1)} \end{pmatrix} \begin{pmatrix} \frac{2}{a+1} \\ -a \end{pmatrix} = \begin{pmatrix} -\frac{a}{a+1} \\ \frac{2a^2}{(a+1)^3} \end{pmatrix}$$

$$\begin{aligned} \alpha_1 &= \operatorname{argmin} \{ (\frac{1}{a+1} - \frac{a}{a+1} \alpha)^2 + \max \{ \frac{2a}{a^2+3a+1} \frac{1}{(a+1)^2} + \frac{2a^2}{(a+1)^3} \alpha, \\ &\quad -a(\frac{2a}{a^2+3a+1} \frac{1}{(a+1)^2} + \frac{2a^2}{(a+1)^3} \alpha) \} \} \\ &= \operatorname{argmin} \{ (\frac{1}{a+1} - \frac{a}{a+1} \alpha)^2 + \frac{2a}{a^2+3a+1} \frac{1}{(a+1)^2} + \frac{2a^2}{(a+1)^3} \alpha \} \\ &= \frac{1}{a(a+1)} \end{aligned}$$

$$\begin{aligned} p_{2k} &= -H_{2k} \nabla f_{2k} \\ &= - \begin{pmatrix} \frac{a^2+2a+2}{2(a^2+a+1)} & \frac{\rho^k}{a(a^2+a+1)} \\ \frac{\rho^k}{a(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k}}{a^2(a^2+a+1)} \end{pmatrix} \begin{pmatrix} 2\rho^k \\ 1 \end{pmatrix} \\ &= - \begin{pmatrix} \frac{(a+1)\rho^k}{a} \\ \frac{2\rho^{2k}}{a^2} \end{pmatrix} \end{aligned}$$

Suppose up to $2k$ th iteration the statement is true. Let us compute the cases for iterations $2k+1$ and $2k+2$.

$$\alpha_{2k} = \operatorname{argmin} \{ (\rho^k - \frac{(a+1)\rho^k}{a} \alpha)^2 - a(\frac{2\rho^{2k}}{a^2+3a+1} - \frac{2\rho^{2k}}{a^2} \alpha) \} = a\rho.$$

$$y_{2k} = \nabla f_{2k+1} - \nabla f_{2k} = (2\frac{\rho^k}{a+1} - 2\rho^k, -(a+1))^T = (-\frac{2a\rho^k}{(a+1)}, -(a+1))^T.$$

$$\begin{aligned} V_{2k} &= I - (p_{2k}^T y_{2k})^{-1} p_{2k} y_{2k}^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{a^2}{2\rho^{2k}(a^2+a+1)} \begin{pmatrix} 2\rho^{2k} & \rho^{k-1} \\ \frac{4\rho^{3k}}{a(a+1)} & \frac{2(a+1)\rho^{2k}}{a^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{a(a+1)^2}{2\rho^k(a^2+a+1)} \\ -\frac{2a\rho^k}{(a+1)(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
H_{2k+1} &= V_{2k} H_{2k} V_{2k}^T + \alpha_{2k} (p_{2k}^T y_{2k})^{-1} (p_{2k} p_{2k}^T) \\
&= \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{a(a+1)^2}{2\rho^k(a^2+a+1)} \\ -\frac{2a\rho^k}{(a+1)(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \begin{pmatrix} \frac{a^2+2a+2}{2(a^2+a+1)} & \frac{\rho^k}{a(a^2+a+1)} \\ \frac{\rho^k}{a(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k}}{a^2(a^2+a+1)} \end{pmatrix} V_{2k}^T \\
&\quad + \alpha_{2k} (p_{2k}^T y_{2k})^{-1} (p_{2k} p_{2k}^T) \\
&= \begin{pmatrix} \frac{a+1}{2(a^2+a+1)} & -\frac{(a+1)\rho^k}{a^2+a+1} \\ -\frac{a\rho^k}{(a+1)(a^2+a+1)} & \frac{2a\rho^{2k}}{(a+1)(a^2+a+1)} \end{pmatrix} \begin{pmatrix} \frac{a+1}{a^2+a+1} & -\frac{2a\rho^k}{(a+1)(a^2+a+1)} \\ -\frac{a(a+1)^2}{2\rho^k(a^2+a+1)} & \frac{a^2}{a^2+a+1} \end{pmatrix} \\
&\quad + a\rho \frac{a^2}{2\rho^{2k}(a^2+a+1)} \begin{pmatrix} \frac{(a+1)^2\rho^{2k}}{a^2} & \frac{2(a+1)\rho^{3k}}{a^3} \\ \frac{2(a+1)\rho^{3k}}{a^3} & \frac{4\rho^{4k}}{a^4} \end{pmatrix} \\
&= \begin{pmatrix} \frac{(a+1)^2}{2(a^2+a+1)} & -\frac{a\rho^k}{a^2+a+1} \\ -\frac{a\rho^k}{a^2+a+1} & \frac{2a^2\rho^{2k}}{(a+1)^2(a^2+a+1)} \end{pmatrix} + \begin{pmatrix} \frac{a^2}{2(a^2+a+1)} & \frac{a\rho^k}{(a+1)(a^2+a+1)} \\ \frac{a\rho^k}{(a+1)(a^2+a+1)} & \frac{2\rho^{2k}}{(a+1)^2(a^2+a+1)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2a^2+2a+1}{2(a^2+a+1)} & -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} \\ -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k}}{(a+1)^2(a^2+a+1)} \end{pmatrix}
\end{aligned}$$

$$p_{2k+1} = -H_{2k+1} \nabla f_{2k+1} = - \begin{pmatrix} \frac{2a^2+2a+1}{2(a^2+a+1)} & -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} \\ -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k}}{(a+1)^2(a^2+a+1)} \end{pmatrix} \begin{pmatrix} \frac{2\rho^k}{a+1} \\ -a \end{pmatrix} = \begin{pmatrix} -\rho^k \\ 2\rho^{2k+1} \end{pmatrix}$$

$$\begin{aligned}
\alpha_{2k+1} &= \operatorname{argmin} \left\{ \left(\frac{\rho^k}{a+1} - \rho^k \alpha \right)^2 - \frac{2\rho^{2k+1}}{a^2+3a+1} + 2\rho^{2k+1} \alpha \right\} \\
&= \operatorname{argmin} \left\{ \rho^{2k} \alpha^2 - \frac{2\rho^{2k}}{a+1} \alpha + 2\rho^{2k+1} \alpha - \frac{2\rho^{2k+1}}{a^2+3a+1} + \frac{\rho^{2k}}{(a+1)^2} \right\} \\
&= \frac{\rho}{a}
\end{aligned}$$

$$y_{2k+1} = \nabla f_{2k+2} - \nabla f_{2k+1} = \begin{pmatrix} -\frac{2\rho^k}{(a+1)^2} \\ a+1 \end{pmatrix}$$

$$\begin{aligned}
V_{2k+1} &= I - (p_{2k+1}^T y_{2k+1})^{-1} p_{2k+1} y_{2k+1}^T \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{(a+1)^2}{2\rho^{2k}(a^2+a+1)} \begin{pmatrix} \frac{2\rho^{2k}}{(a+1)^2} & -(a+1)\rho^k \\ -\frac{4a\rho^{3k}}{(a+1)^4} & \frac{2a\rho^{2k}}{a+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a^2+a}{a^2+a+1} & \frac{(a+1)^3}{2\rho^k(a^2+a+1)} \\ \frac{2\rho^{k+1}}{a^2+a+1} & \frac{1}{a^2+a+1} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
H_{2k+2} &= V_{2k+1} H_{2k+1} V_{2k+1}^T + \alpha_{2k+1} (p_{2k+1}^T y_{2k+1})^{-1} p_{2k+1} p_{2k+1}^T \\
&= \begin{pmatrix} \frac{a^2+a}{a^2+a+1} & \frac{(a+1)^3}{2\rho^k(a^2+a+1)} \\ \frac{2\rho^{k+1}}{a^2+a+1} & \frac{1}{a^2+a+1} \end{pmatrix} \begin{pmatrix} \frac{2a^2+2a+1}{2(a^2+a+1)} & -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} \\ -\frac{a^2\rho^k}{(a+1)(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k}}{(a+1)^2(a^2+a+1)} \end{pmatrix} V_{2k+1}^T \\
&\quad + \frac{1}{2\rho^{2k}(a^2+a+1)} \begin{pmatrix} \rho^{2k} & -2\rho^{3k+1} \\ -2\rho^{3k+1} & 4\rho^{4k+2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a^2+a}{2(a^2+a+1)} & \frac{\rho^k}{a^2+a+1} \\ \frac{\rho^{k+1}}{a^2+a+1} & \frac{2\rho^{2k}}{(a+1)^3(a^2+a+1)} \end{pmatrix} \begin{pmatrix} \frac{a^2+a}{a^2+a+1} & \frac{2\rho^{k+1}}{a^2+a+1} \\ \frac{(a+1)^3}{2\rho^k(a^2+a+1)} & \frac{1}{a^2+a+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{2(a^2+a+1)} & -\frac{\rho^{k+1}}{a^2+a+1} \\ -\frac{\rho^{k+1}}{a^2+a+1} & \frac{2\rho^{2k+2}}{a^2+a+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{(a+1)^2}{2(a^2+a+1)} & \frac{\rho^k}{(a+1)(a^2+a+1)} \\ \frac{\rho^k}{(a+1)(a^2+a+1)} & \frac{2\rho^{2k}}{(a+1)^4(a^2+a+1)} \end{pmatrix} + \begin{pmatrix} \frac{1}{2(a^2+a+1)} & -\frac{\rho^{k+1}}{a^2+a+1} \\ -\frac{\rho^{k+1}}{a^2+a+1} & \frac{2\rho^{2k+2}}{a^2+a+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a^2+2a+2}{2(a^2+a+1)} & \frac{\rho^{k+1}}{a(a^2+a+1)} \\ \frac{\rho^{k+1}}{a(a^2+a+1)} & \frac{2(a^2+1)\rho^{2k+2}}{a^2(a^2+a+1)} \end{pmatrix}
\end{aligned}$$

The proposition follows. \square

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